

## Homework 2

(Due Thursday 15th February 2007)

1. **The astronaut-on-a-bungee-cord problem :** It is the near future, and (after a couple of close calls!) NASA lawyers insist that any astronauts making a space-walk around the International Space Station must be tethered to the space station via an elastic bungee cord. However, in order not to interfere with the astronauts' operations, the decision is made that these bungee cords will be "weak springs". In this question, you will examine the dynamics of an astronaut tethered via a weak bungee cord and come to the not-so-obvious conclusion that the lawyers have made a horrible mistake!

- (a) Let the astronaut and space-station have masses  $m_1$  and  $m_2$ , respectively. Also, assume that the space-station is *almost* on a circular orbit about the center of the Earth with radius  $R_0$ , and that the astronaut is (at least initially) close to the space-station (i.e., the separation between the astronaut and the space-station is  $\ll R_0$ ). By performing a 1st-order perturbation analysis very similar to that shown in class, show that the equations of motion for astronaut are:

$$\frac{d^2x_1}{dt^2} - 3\Omega_0^2x_1 - 2\Omega_0\frac{dy_1}{dt} - \frac{k}{m_1}(x_2 - x_1) = 0 \quad (1)$$

$$\frac{d^2y_1}{dt^2} + 2\Omega_0\frac{dx_1}{dt} - \frac{k}{m_1}(y_2 - y_1) = 0 \quad (2)$$

where  $\Omega_0 \equiv (GM/R_0^3)^{1/2}$ ,  $k$  is the spring constant of the bungee cord (i.e., the force exerted by the bungee cord has magnitude  $F = k|\mathbf{r}_2 - \mathbf{r}_1|$ ),  $(x_1, y_1)$  are the cartesian coordinates of the astronauts relative to the unperturbed circular orbit, and  $(x_2, y_2)$  are the cartesian coordinates of the spacestation relative to the unperturbed circular orbit.

- (b) With just one sentence of justification, write down the equations of motion for the space-station. Noting that the the space-station is much more massive that the astronaut ( $m_2 \gg m_1$ ), go on to show that

$$\frac{d^2}{dt^2}(\Delta x) - \left(3\Omega_0^2 - \frac{k}{m_1}\right)\Delta x - 2\Omega_0\frac{d}{dt}(\Delta y) = 0 \quad (3)$$

$$\frac{d^2}{dt^2}(\Delta y) + 2\Omega_0\frac{d}{dt}(\Delta x) + \frac{k}{m_1}\Delta y = 0 \quad (4)$$

- (c) Suppose that the bungee cord is “strong”, i.e., has a “large” spring constant (in some sense that you should make more precise in your answer). Simplify and solve the equations of motion. Describe in words the motion of the astronaut relative to the space-station in this case.
- (d) Now consider the case where the bungee rope is not necessarily strong. Combine eqns (3) and (4) into a single fourth-order differential equation for  $\Delta x$  and show that the characteristic equation of this differential equation (i.e., the values of  $\alpha$  which satisfy the differential equation when one substitutes  $\Delta x \propto e^{\alpha t}$ ) is

$$\alpha^4 + \left( \Omega_0^2 + \frac{2k}{m_1} \right) \alpha^2 - \frac{k}{m_1} \left( 3\Omega_0^2 - \frac{k}{m_1} \right) = 0 \quad (5)$$

- (e) Explain why the solution of the equations of motion will “blow up” (i.e., become arbitrarily large as time goes on) if the characteristic equation has any real positive roots.
- (f) Aided by a graph, or otherwise, show that the characteristic equation must have real-positive roots if  $k < 3m_1\Omega_0^2$ . On the basis of your answers above, explain in words the motion of the astronaut relative to the space-station if the bungee cord is “weak”, i.e.,  $k < 3m_1\Omega_0^2$ .

*[Footnote : The instability discussed in this question is very closely related to an instability that is of profound importance for accretion disks around black holes or young stars, the so-called magnetorotational instability. In that case, the astronaut and the space-station are replaced by elements of ionized gas, and the bungee cord is replaced by magnetic tension. The resulting instability drives powerful turbulence within the accretion disk, redistributing angular momentum and allowing accretion to actually proceed.]*