Sources of Gravitational Radiation

As direct detection of gravitational radiation draws nearer, it is useful to consider what such detections will teach us about the universe. The first such detection, of course, will be of immediate significance because it will be a direct confirmation of a dramatic prediction of general relativity: to paraphrase John Wheeler, that spacetime tells sources how to move, and moving sources tell spacetime how to ripple.

Beyond this first detection, gravitational wave detections will pass into the realm of astronomy, allowing new windows onto some of the most dynamic phenomena in the universe. These include merging neutron stars and black holes, supernova explosions, and possibly echoes from the very early history of the universe as a whole. They are also anticipated to provide the cleanest tests of predictions of general relativity in the realm of strong gravity.

However, there are important differences from standard astronomy. In electromagnetic observations, in every waveband there are sources so strong that they can be detected without knowing anything about the source. You don't need to understand nuclear fusion in order to see the Sun! In contrast, as we will see, most of the expected sources of gravitational radiation are so weak that sophisticated statistical techniques are required to detect them at all. These techniques involve matching templates of expected waveforms against the observed data stream. Maximum sensitivity therefore requires a certain understanding of what the sources look like, hence of the characteristics of those sources. In addition, when detections occur, it will be important to put them into an astrophysical context so that the implications of the discoveries are evident.

It is useful to remember that historically the most interesting sources discovered with a new telescope or satellite have often been unexpected, and this is also possible with gravitational radiation. However, you can't sell a large project by appealing entirely to the unknown, so we should at least describe what we *can* imagine at this point!

Before discussing types of sources, though, we need to have some general perspective on how gravitational radiation is generated and how strong it is. We will begin by discussing radiation in a general context.

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ,ϕ) is $A(r,\theta,\phi)$, the flux through a spherical surface at r is $F(r,\theta,\phi)\propto A^2(r,\theta,\phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r,\theta,\phi)=A(r)$, this means that the luminosity at a distance r is $L(r)\propto A^2(r)4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r)\propto 1/r^2$, implying that $L(r)\propto 1/r^2$ and hence no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

We can now explore what types of variation will produce radiation. We'll start with electromagnetic radiation, and expand in moments. For a charge density $\rho_e(\mathbf{r})$, the monopole moment is $\int \rho_e(\mathbf{r}) d^3r$. This is simply the total charge Q, which cannot vary, hence there is no electromagnetic monopolar radiation. The next static moment is the dipole moment, $\int \rho_e(\mathbf{r})\mathbf{r} d^3r$. There is no applicable conservation law, so electric dipole radiation is possible. One can also look at the variation of currents. The lowest order such variation (the "magnetic dipole") is $\int \rho_e(\mathbf{r})\mathbf{r}\times\mathbf{v}(\mathbf{r}) d^3r$. Once again this can vary, so magnetic dipole radiation is possible. The lower order moments will typically dominate the field unless their variation is reduced or eliminated by some special symmetry.

Now consider gravitational radiation. Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r}) d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r})\mathbf{r} d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The equivalent of the magnetic dipolar moment is $\int \rho(\mathbf{r})\mathbf{r}\times\mathbf{v}(\mathbf{r}) d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r}) r_i r_j d^3r$. This is not conserved, therefore there can be quadrupolar gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, hence it does not produce radiation. No matter how violent an explosion or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains axisymmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (the most interesting of which would be a background of gravitational waves from the early universe). We will focus on binaries, which are the only observed sources which will definitely give amplitudes in the detectable range.

We start out with some order of magnitude estimates. What is the approximate expression for the dimensionless amplitude h of a gravitational wave, a distance r from a source? That is, we would like to know the fractional amount by which a detector is stretched or squeezed when a gravitational wave passes by.

We argued that the lowest order radiation had to be quadrupolar, and hence depend on the quadrupole moment I. This moment is $I_{ij} = \int \rho r_i r_j d^3x$, so it has dimensions MR^2, where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r \; . \tag{1}$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective "units", but we can't, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can't emit anything. Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \; . \tag{2}$$

Now what? To get back to physical units we have to restore factors of G and c. It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \; . \tag{3}$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless M and R are large, the system is changing fast, and r is small, the metric perturbation is minuscule.

Let's make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2/M$, the semimajor axis is a, and the orbital frequency Ω is therefore given by $\Omega^2 a^3 = M$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G/c^4)(1/r)(\mu M/a) \; . \tag{4}$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal mass system

$$h \approx 10^{-22}\left(\frac{M}{2.8M_\odot}\right)^{5/3}\left(\frac{0.01\text{ sec}}{P}\right)^{2/3}\left(\frac{100\text{ Mpc}}{r}\right) \; , \tag{5}$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That's why it is so challenging to detect these systems!

Remarkably, though, the flux of energy is *not* tiny. To see this, let's calculate the flux given some dimensionless amplitude h. The flux has to be proportional to the square of the amplitude and also the square of the frequency f: $F \sim h^2 f^2$. This currently has units of time squared, but the physical units of flux are energy per time per area. Replacing factors of G and c, we find that the flux is

$$F \sim (c^3/G)h^2 f^2 \ . \tag{6}$$

Now the prefactor is *enormous*! For the double neutron star system above, with $h \sim 10^{-22}$ and $f \sim 100$ Hz, this gives a flux of a few hundredths of an erg cm^{-2} s^{-1}. For comparison, the flux from Sirius, the brightest star in the night sky, is about 10^{-4} erg cm^{-2} s^{-1}! That means that if you could somehow absorb gravitational radiation perfectly with your eyes, you would find untold numbers of sources brighter than every star except the Sun. What this really implies, of course, is that gravitational radiation interacts *very* weakly with matter, which again means that it is mighty challenging to detect.

We now consider binary systems more specifically. These obviously have a large and varying quadrupole moment, and have the additional advantage that we actually know that gravitational radiation is emitted from them in the expected quantities (based on observations of double neutron star binaries). The characteristics of the gravitational waves from binaries, and what we could learn from them, depend on the nature of the objects in those binaries. We will therefore start with some general concepts and then discuss individual types of binaries.

First, let's get an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, the density, we can say that the maximum frequency involving an object of density ρ is $f_{\rm max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave through the object can't be greater than $\sim (G\rho)^{1/2}$. Therefore, this is a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-3} - 10^{-6}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1 - 10$ Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000 - 2000$ Hz, and that for black holes the limit depends inversely on mass (and also spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is on the order of $10^4(M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at lower frequencies.

Do we have evidence that these formulae actually work? Yes! Nature has been kind enough to provide us with the perfect test sources: binary neutron stars. Several such systems are known, all of which have binary separations orders of magnitude greater than the size of a neutron star, so the lowest order formulae should work. Indeed, predictions about the rate of change of the semimajor axes of double neutron star binaries have been verified to better than 0.1% in a few cases. The de/dt predictions will be much tougher to verify, though. The reason for the difference is that de/dt has to be measured by determining the eccentricity orbit by orbit, whereas da/dt has a manifestation in the total phase of the binary, so it accumulates quadratically with time. These systems provide really spectacular verification of general relativity in weak gravity. In particular, in late 2003 a double pulsar system was detected, that in addition has the shortest expected time to merger of any known system (only about 80 million years). Having two pulsars means that extra quantities can be measured (such as the relative motion, which gives us the mass ratio), and in fact the system is now dramatically overconstrained (more things measured than there are parameters in the theory). The tests of GR by observations of binary neutron star systems deservedly resulted in the 1993 Nobel Prize in physics going to Hulse and Taylor, who discovered the first such binary.

One interesting effect that emerges from the higher-order studies of binary inspirals is that gravitational radiation carries away net linear momentum, hence the center of mass of the system moves in an ever-widening spiral. We can understand this as follows (following an idea of Alan Wiseman). In an unequal-mass binary, the lower-mass object moves faster. As the speed in orbit becomes relativistic, the gravitational radiation from each object becomes beamed, with the lower-mass object producing more beaming because it moves faster. Therefore, at any given instant, there is a net kick against the direction of motion of the lower-mass object. If the binary were forced to move in a perfect circle, the center of mass of the system would simply go in a circle as well. However, because in reality the orbit is a tight and diminishing spiral, the recoil becomes stronger with time and the center of mass moves in an expanding spiral. Note that by symmetry, equal-mass nonspinning black holes can never produce a linear momentum kick, and that if the mass ratio is gigantic the fractional energy release is small and therefore so is the kick. For nonspinning holes, the optimal ratio for a kick is about 2.6.

This process is potentially important astrophysically because if the final merged remnant of a black hole inspiral is moving very rapidly, it could be kicked out of its host stellar system, with possibly interesting implications for supermassive black holes and hierarchical merging. There have therefore been a number of calculations of the expected kick. It has turned out that these are very challenging. The primary reason is that most of the action is near the end, when the black holes are close to each other and simple approximations to the orbit are inaccurate. Fortunately, reliable and fully numerical calculations are just now becoming available; see astro-ph/0603204 for the first such calculation, with non-spinning black holes in a 1.5:1 mass ratio.