ASTR498E High Energy Astrophysics (Spring 2006)

Homework 1

Due date; 16th February 2006

1. **Degeneracy pressure from a cold electron gas :** In this question, you will derive the full formula quoted in class for the degeneracy pressure from a cold electron gas. A couple of important concepts will be introduced along the way.

   (a) Consider electrons in a cubical box with side length L. With the aid of an appropriate diagram, briefly explain why the three components of all of the electrons' momenta must satisfy:

   $$p_x = \frac{hn_x}{2L} \qquad p_y = \frac{hn_y}{2L} \qquad p_z = \frac{hn_z}{2L}, \tag{1}$$

   $$n_x, n_y, n_z = 1, 2, 3, \ldots \tag{2}$$

   Thus, explain why the number of electron quantum states in the momentum range $p_x \to p_x + \delta p_x$, $p_y \to p_y + \delta p_y$, $p_z \to p_z + \delta p_z$, the so-called "density of states", is given by

   $$\frac{16L^3}{h^3}\delta p_x \delta p_y \delta p_z \tag{3}$$

   (b) If $p = |\mathbf{p}|$ is the magnitude of electron momenta, show that the number of quantum states with momentum in the range $p \to p + \delta p$ per unit volume of the cube is given by

   $$n(p)\,dp = \frac{8\pi p^2}{h^3}\,dp \tag{4}$$

   [Hint: in addition to using the density of states results from (a), you will have to consider the momentum-space volume of the region $p \to p + \delta p$].

   (c) Suppose that the total number of electrons per unit volume is n_e. Furthermore, suppose that these electrons are "cold" so that the fill the available quantum states in order of increasing energy (i.e., they fill the states "from the bottom up"). Let the maximum electron momentum (the so-called Fermi momentum) be denoted by p_F. Write down an integral relating n_e, $n(p)$ and p_F. By evaluating this integral, show that the Fermi momentum is given by

   $$p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}. \tag{5}$$

   (d) A general formula linking the pressure exerted by an (isotropic) gas to the number of particles that have momenta in the range $p \to p + \delta p$, $n_e(p)$ is

   $$P = \frac{1}{3}\int_0^\infty p v(p) n(p)\,dp, \tag{6}$$

where v is the velocity of a particle with momentum p (this formula is ready derived by considering the motion of particles bouncing off the wall of the box). Using this integral, derive the relationship between pressure P and total electron number density n_e for the case of non-relativistic electrons ($p = mv$) and highly relativistic electrons ($v \approx c$).

(e) Noting that the actual density of the matter in the box is given by $\rho = \mu m_p n_e$ (same μ as discussed in class), express your answers from part (d) in terms of the actual mass density. This should agree with the expressions quoted in class.

(f) White dwarf stars are spheres, not cubical boxes. One might worry that the spherical nature of the star would affect the forms of the wave-functions and the spacing of the quantum states. Qualitatively explain why this does not matter in this analysis.

2. **Cold boson stars :** Electrons are *fermions*, i.e., particles with a half-integer spin which cannot share a quantum state with an identical fermion due to the Paulii exclusion principle. The other type of fundamental particle are *Bosons*; these have integer values of spin (in units of $\hbar$) and do not obey the exclusion principle. In fact, there is an enhanced probability of finding two bosons in the same quantum state. In this question we will explore the properties of a star consisting of cold, fully-degenerate hypothetical bosons of mass m.

   (a) Given the assumptions listed in the last sentence above, explain why the momentum of every particle in the star will be approximately $p \sim h/R$, where R is the radius of the star.

   (b) Using the usual expression relating the pressure to the momentum distribution given in question 1-(d), derive approximate expressions for the pressure as a function of density ρ, R, and m for both the non-relativistic and relativistic cases.

   (c) By approximating the equation of hydrostatic equilibrium to its simplified algebraic form, use the expression for the non-relativistic pressure above to show that

   $$R \sim \frac{h^2}{GMm^2}, \tag{7}$$

   where M is the total mass of the boson star.

   (d) To avoid total collapse to a black hole, any object of mass M must have a radius larger than GM/c^2. Use this fact to show that the maximum possible mass $M_{\rm max}$ of a boson star made of bosons with mass m is

   $$M_{\rm max} \sim \frac{4\pi^2 m_{\rm Pl}^2}{m}, \tag{8}$$

   where $m_{\rm Pl} = \sqrt{hc/2\pi G}$ is known as the *Planck Mass*. Show that the bosons are becoming relativistic at about this same mass.

   (e) Evaluate this maximum mass in the case of extremely "light" degenerate bosons which have a mass $m = 10^{-20} m_p$ (where m_p is the mass of a proton). Evaluate your answer in solar masses.