

Homework 4

Due date; Thursday 27th April 2006

1. **Black holes and the area theorem** : Consider an uncharged black hole with a mass M and a scaled spin parameter of a (such that $a = 1$ corresponds to a black hole that is rotating at the maximum possible rate). The total energy of this black hole is $E = Mc^2$. Extensive investigation by Stephen Hawking and others showed black holes also possess an *entropy* which is proportional to the *area* of the event horizon,

$$A = 8\pi M^2 \left(1 + \sqrt{1 - a^2}\right). \quad (1)$$

Hawking showed that no allowable physical process can decrease the event horizon area of a black hole. This result generalizes to an interacting collection of black holes — the sum of the areas of the event horizons cannot decrease. This is known as the area theorem and is closely related to the second law of thermodynamics (which states that the entropy of a closed system can never decrease).

- (a) Suppose that we start with a single maximally spinning ($a = 1$) black hole of mass M and, by some process, we extract angular momentum and hence spin it down until it stops rotating ($a = 0$). In the process of doing this, we extract the rotational energy of the black hole. By writing the mass of the initial and final states in terms of the event horizon area, show that the maximum possible energy that can be extracted from the spinning black hole is

$$\Delta E = \left(1 - \frac{1}{\sqrt{2}}\right) Mc^2. \quad (2)$$

- (b) Now consider a slowly rotating black hole ($a \ll 1$) which is also being spun down. Use the area theorem (and a Taylor expansion) to show that one may, in principal, extract up to $E \approx Mc^2 a^2 / 8$ of energy from this spinning black hole.
- (c) Suppose two non-rotating black holes with mass M_1 and M_2 merge to form a single non-rotating black hole. Use the area theorem to deduce the *maximum* amount of energy that can be carried away from this system by gravitational radiation.

2. **Some properties of the intracluster medium in galaxy clusters** : In this question, we examine a (very) simple model for the intracluster medium (ICM) of a galaxy cluster. We assume spherical symmetry and, further, assume that the gravitational potential of the cluster is completely dominated by the non-baryonic dark matter which possesses a density $\rho_{\text{dm}}(r)$. We then make the (reasonable) assumption that the ICM is in hydrostatic equilibrium, leading to the equation

$$\frac{dP}{dr} = -\rho \frac{GM_{\text{dm}}(< r)}{r^2}, \quad (3)$$

where ρ and P are the density and pressure of the ICM, and $M_{\text{dm}}(< r)$ is the dark matter mass enclosed within a sphere of radius r .

- (a) Suppose that X-ray measurements reveal that the ICM is isothermal (i.e., has the same temperature and hence sound speed, c_s , everywhere) so that $P = 3c_s^2\rho/5$ (where the factor $3/5$ comes from the usual ratio of specific heats for a monatomic gas). Further suppose that the measurements reveal an ICM density profile

$$\rho(r) = \rho_0 \left(\frac{r_0}{r} \right), \quad (4)$$

where r_0 is some reference radius and ρ_0 is the ICM density at that radius. Use this information to deduce that the dark matter density is

$$\rho_{\text{dm}}(r) = \frac{c_s^2}{4\pi r^2 G \gamma} \quad (5)$$

- (b) If the ICM sound speed is 700 km s^{-1} and its density at 100 kpc is $1.5 \times 10^{-26} \text{ g cm}^{-3}$, determine the ratio of the ICM gas mass enclosed within a 100 kpc radius sphere to the dark matter mass within the same sphere. [You will need to know that $1 \text{ kpc} \approx 3 \times 10^{21} \text{ cm}$. Be careful of units!]
- (c) How does this compare with the overall baryonic-to-dark-matter ratio in the Universe? If different, discuss possible explanations for the difference.