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LORENTZ AND GRAVITATIONAL RESONANCES ON CIRCUMPLANETARY PARTICLES

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ABSTRACT

Micron-sized circumplanetary dust particles are subject to various non-gravitational perturbations, principally solar radiation pressure and electromagnetic forces, which are typically a few percent as strong as the planetary gravity. Individually, these perturbations can cause some orbital evolution, but when the perturbations act in concert the excursions can be much larger. We demonstrate this effect for a single example, the coupling between resonances and drag forces. Throughout this work, we emphasize the parallels between satellite-satellite gravitational resonances and their electromagnetic counterparts (Lorentz resonances).

INTRODUCTION

A dynamical system typically has a set of natural frequencies at which it can rotate or vibrate. When such a system is forced at one of these natural frequencies (or a multiple of it), the amplitude of oscillations grows as a result of the cumulative effect of in-phase perturbations; the system is said to be in resonance. A child on a swing provides a familiar example of a resonant system. If the swing (initially at rest) is pushed at an arbitrary frequency or at random times, the amplitude of oscillation is likely to remain small; if, however, the swing is pushed once per period, the oscillation amplitude will grow quite large. In an entirely similar manner, charged dust grains oscillate wildly near the locations of "Lorentz resonances" which occur at those positions where the electromagnetic force sensed by an orbiting particle (and arising from a planet's spinning magnetic field) has a component that matches a natural frequency of the orbit /1/. The abrupt vertical expansion of the jovian ring into its halo and the disappearance of the halo itself /1,2/ have been ascribed to the action of these Lorentz resonances on orbiting dust grains.

Gravitational resonances occur when the orbital periods of two objects are nearly a simple ratio of integers. Many features in the main saturnian ring system have been successfully attributed to gravitational resonances with exterior satellites. For example, the 2:1 resonance with Mimas defines the inner edge of the Cassini division, which divides the A and B rings, while the sharp outer edge of the A ring occurs at a 7:6 resonance with the moon Janus. Satellites themselves are often found in resonances with one another; examples include the saturnian pairs Enceladus/Dione, Titan/Hyperion and Mimas/Tethys, as well as the jovian triple Io/Europa/Ganymede (see /3/ for a qualitative physical description of these gravitational resonances).

In this paper we wish to illustrate how resonances couple with drag forces. This idea is not new; indeed it has been extensively studied in the context of satellite evolution where tidal effects from the central body create small drags on satellite orbits. This problem has been thoroughly treated using Hamiltonian mechanics (see *e.g.*, /4/). The purpose of the current paper is twofold. First, we wish to draw parallels between the extensively studied satellite (gravitational) resonances and their less well known relatives, Lorentz resonances. Secondly, we will reproduce some results of the Hamiltonian theory using the Lagrangian orbital perturbation equations /5/, which are written in terms of the orbital elements. The latter quantities provide a physically meaningful description of an orbit; for orbits confined to a particular plane, the semimajor axis a, the eccentricity e, and the longitude of pericenter $\tilde{\omega}$ are sufficient. These three elements, respectively, describe the instantaneous size, shape, and orientation of an elliptical orbit; the Lagrangian equations that describe the time rate of change of such orbital elements are well suited to visualizing the results of orbital perturbations. The advantage of our approach is its simplicity: many non-intuitive effects of resonances, such as resonant trapping and jumps, will be elucidated.

RESONANCE EQUATIONS

The problem of determining the perturbing effects of one satellite on another is fundamental to celestial mechanics and has been studied for centuries. It is not solvable in closed form, but an approximate solution can be developed as a power series of small quantities. The typical procedure (*cf.* /5/, p. 339) is as follows. First, one evaluates the disturbing function, defined as the negative of the perturbing satellite's potential, at the position of the perturbed particle. Next, the disturbing function is written in terms of the orbital elements; this step requires complicated power series expansions in eccentricities, inclinations, and the semimajor axis ratio. Finally the changes to the orbital elements can be calculated with the potential form of Lagrange's planetary equations (/5/, p. 336) which relate the time rates of change of the orbital elements to derivatives of the disturbing function and to instantaneous values of the elements themselves.

We proceed in a similar manner for Lorentz resonances. Because the Lorentz force due to a magnetic field cannot be derived from a potential, we must calculate the electromagnetic force arising from an arbitrary magnetic field and express it in terms of orbital elements, an arduous task which requires power series expansions in the particle's eccentricity and inclination. These forces are then inserted into an alternate form of Lagrange's planetary equations (/5/, p. 327). The results of this calculation yield, as above, expressions for time derivatives of the orbital elements which are functions of the instantaneous values of these elements. We plan to submit the details of this calculation for publication in *Icarus.*

In both of the above derivations, secular terms (*i.e.*, those that do not depend on satellite longitudes) as well as periodic terms (with longitude dependence) appear. Secular terms are ubiquitous, whereas periodic terms, over long times, average to zero at all but a few resonant locations. In this paper we focus on one of these locations as an example: the 2:1 (first-order) eccentricity resonance. Near this location, the resonant argument ϕ is given by:

$$\phi = \lambda - 2\lambda' + \tilde{\omega}, \tag{1}$$

where λ and λ' are the longitudes of the perturbee and perturber, respectively. At the resonant location (defined by $\phi = 0$ - see figure 1), the perturbed body completes approximately two orbits for every one cycle of the perturbing force (the period of an exterior satellite in the gravitational case or the planetary spin period for Lorentz resonances). We ignore all periodic terms with different frequency dependencies (since they average to zero), and the secular perturbations (which are small compared to the strong 2:1 resonant terms).

The orbital elements most strongly affected by such a resonance are the abovementioned a, e, and $\tilde{\omega}$. Instead of the semimajor axis a, we use the unperturbed orbital mean motion $n \sim \dot{\lambda}$, which is related to the semimajor axis via $n^2a^3 = GM_p$, where G is the gravitational constant and M_p is the planetary mass (/5/, p. 131). Writing out the Lagrange perturbation equations to lowest order in eccentricity and inclination, we find that the effects of both the gravitational and Lorentz versions of the 2:1 first-order eccentricity resonance can be represented by a set of equations of the following form:

$$\frac{dn}{dt} = -3en^2\beta \sin\phi \tag{2a}$$

$$\frac{de}{dt} = -nA_1\beta\sin\phi \tag{2b}$$

$$\frac{d\tilde{\omega}}{dt} = -\frac{nA_2\beta}{e}\cos\phi. \tag{2c}$$

Here t is time, β (always positive) measures the appropriate resonance strength and the A_i are constants. The quantity β is a complicated function of the semimajor axis ratio which must be expanded as a power series; across the small distance over which the resonance exerts its influence, however, β can be treated as a constant. In the gravitational case, β is first order in the satellite/planet mass ratio and $A_1 = A_2 = 1$. In the Lorentz case, β depends on the particle's charge-to-mass ratio, distance from the planet, and the magnetic field strength. For first-order electromagnetic resonances, $A_1 = A_2 = n/n' - 1$, so the 2:1 resonance, like gravity, has $A_1 = A_2 \approx 1$. The dominant contribution to this resonance comes from the g_{32} component of the magnetic field (a non-symmetric octupole term - see /2/ which gives values for the giant planets).

Although we have specialized equations (2a-c) to the 2:1 eccentricity resonance, the form of the equations for other first-order eccentricity resonances (2:3, 3:4, 1:2 etc.) is entirely similar - only the parameters β and the A_i need to be changed. First-order inclination resonances (which exist for Lorentz forces but not for satellite gravity) and higher-order resonances are also not too different. Accordingly, the general behavior discussed below for the 2:1 eccentricity resonance actually applies to a wide variety of other types of resonances as well; that is to say, the trapping and jumps discussed below are general phenomena.

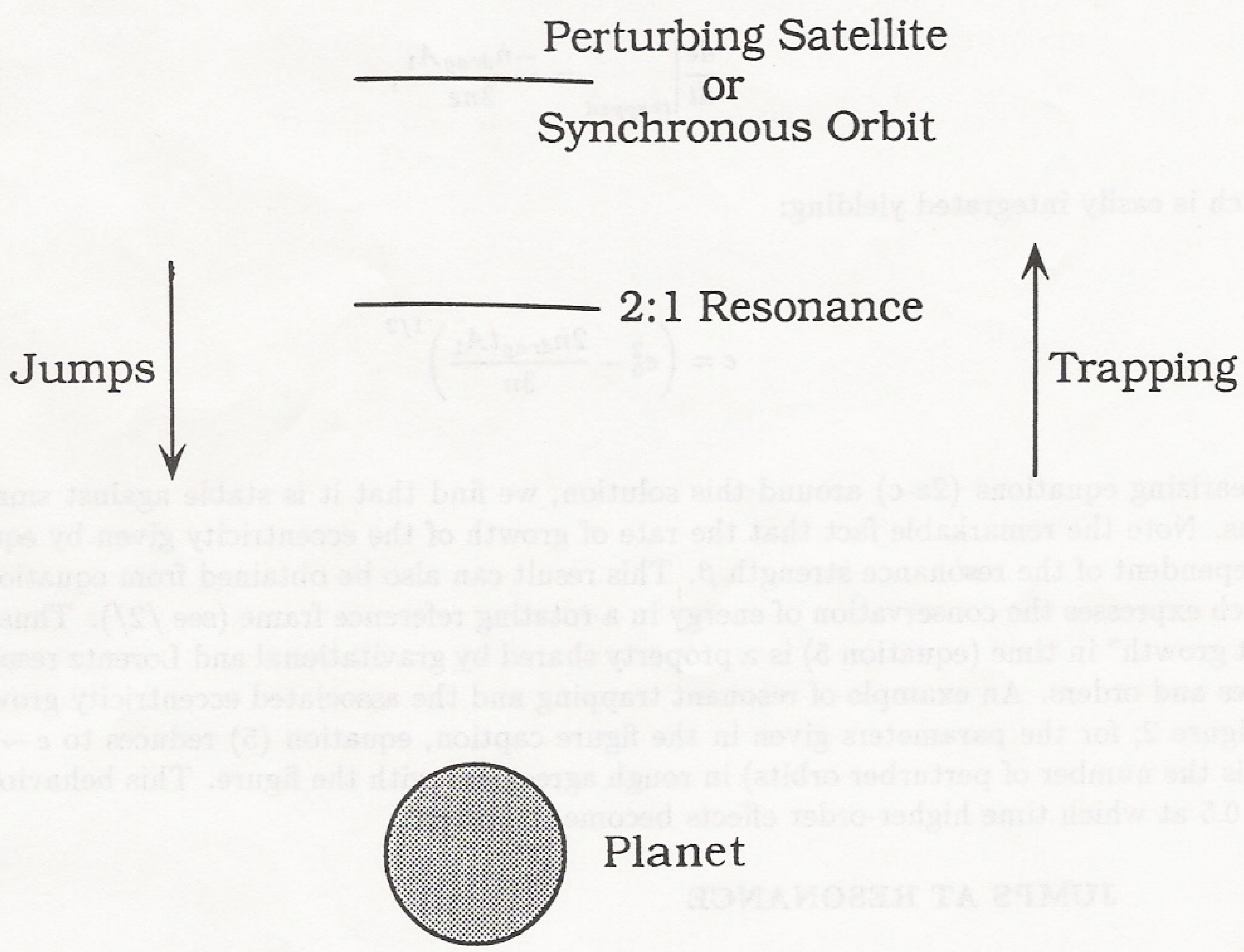

Fig. 1. Schematic diagram showing the central planet, the orbiting dust grain, and the 2:1 resonance. The outermost line represents the location of the perturbing satellite (for a gravitational resonance) or of synchronous orbit (for a Lorentz resonance). A grain drifting through a first-order resonance toward this location may become trapped while one drifting away from it will experience a jump.

DRAG FORCES

Several drag forces operate in the magnetospheres of the giant planets. Most large satellites are driven slowly outward by tidal forces from the primary while small particles are affected by a host of processes /6/ including plasma, atmospheric, and Poynting-Robertson drags which, for dust grains, operate much more rapidly than tidal evolution. Because drag forces are typically much smaller than many other orbital perturbations, their effects on most orbital elements can often be ignored. Unlike most other perturbations, however, drag forces systematically affect an orbit's energy and therefore its size and mean motion. Furthermore, because of the limited radial extent of the resonance zone, we can approximate the functional form of the drag rate in this region by a simple constant $\dot{n}_{drag}$. The inclusion of drag forces requires that we replace equation (2a) with

$$\frac{dn}{dt} = -3en^2\beta\sin\phi + \dot{n}_{drag}. \tag{3}$$

RESONANCE TRAPPING

When $\dot{n}_{drag} < 0$, orbits evolve outward: near the 2:1 resonance, this evolution is toward the perturbing satellite (in the case of gravity) or toward synchronous orbit (in the Lorentz case). For this type of evolution, resonance trapping, in which the evolution in mean motion ceases, is possible (figure 1). Clearly trapping can occur only if the first term in equation (3) is equal and opposite to the second for some ϕ. Solving equation (3) for $\sin\phi$ in this case and substituting into equation (2b), we find

$$\left.\frac{de}{dt}\right|_{trapped} = \frac{-\dot{n}_{drag}A_1}{3ne}, \tag{4}$$

which is easily integrated yielding:

$$e = \left(e_0^2 - \frac{2\dot{n}_{drag}tA_1}{3n}\right)^{1/2}. \tag{5}$$

Linearizing equations (2a-c) around this solution, we find that it is stable against small perturbations. Note the remarkable fact that the rate of growth of the eccentricity given by equation (5) is independent of the resonance strength β. This result can also be obtained from equation (7) below, which expresses the conservation of energy in a rotating reference frame (see /2/). Thus the "square root growth" in time (equation 5) is a property shared by gravitational and Lorentz resonances of all types and orders. An example of resonant trapping and the associated eccentricity growth is shown in figure 2; for the parameters given in the figure caption, equation (5) reduces to $e \sim 0.00145N^{1/2}$ (N is the number of perturber orbits) in rough agreement with the figure. This behavior holds until $e \sim 0.5$ at which time higher-order effects become important.

JUMPS AT RESONANCE

When $\dot{n}_{drag} > 0$, inner orbits evolve away from the perturbing satellite (or from synchronous orbit). In this case trapping for low eccentricities is not possible as can be seen from equation (5) which implies that eccentricity becomes imaginary! Instead we shall find a different behavior at the resonant location.

Because drag forces are so small, the first term in equation (3) is usually far greater than the second; this fact allows us to obtain an adiabatic invariant. Ignoring the drag term for the moment, we divide equation (2a) by equation (2b) and find

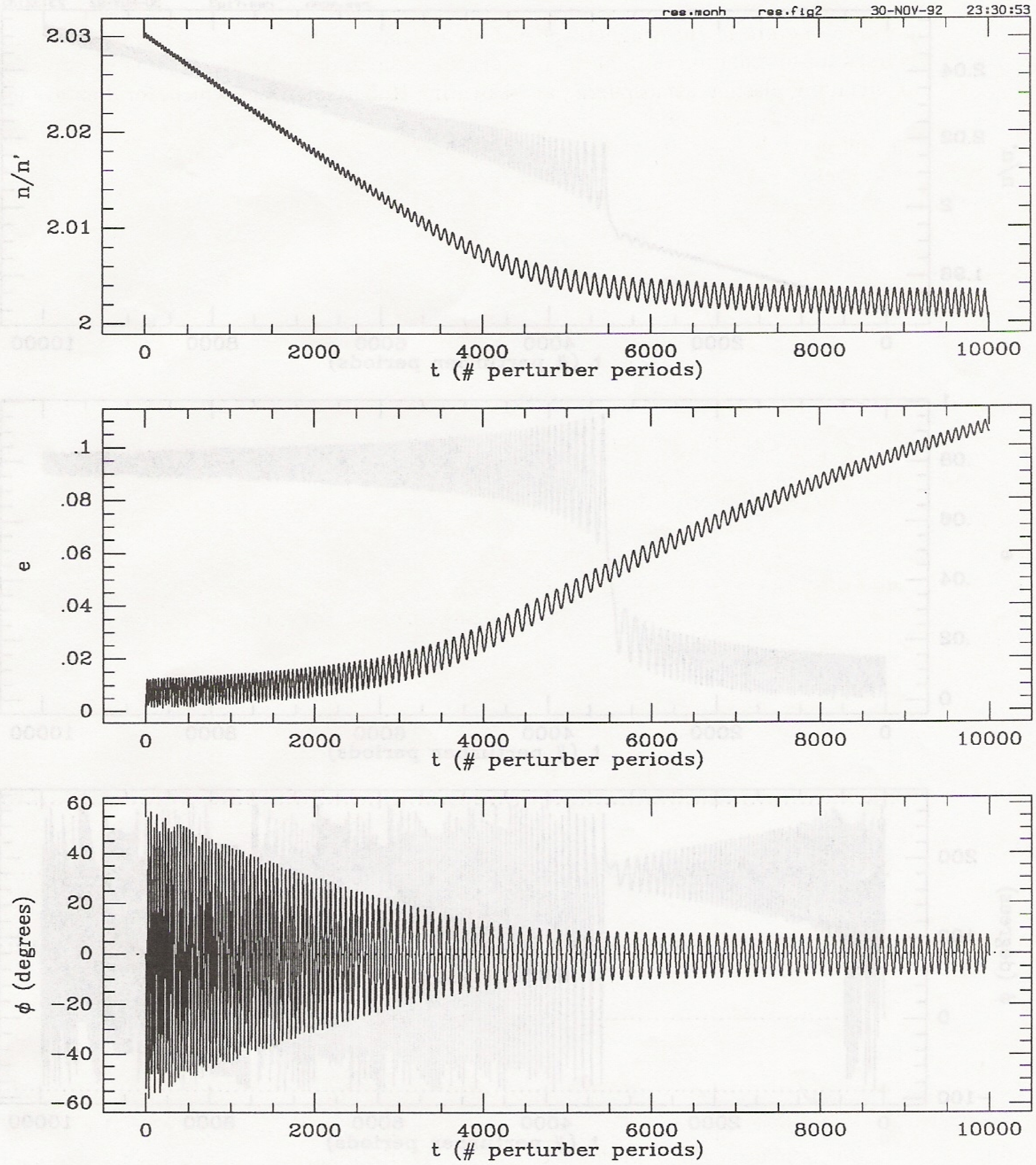

Fig. 2. RESONANCE TRAPPING: A plot of the orbital evolution determined by equations (2b,c and 3) for physically realistic parameters $\beta = 10^{-4}; A_1 = A_2 = 1; \dot{n}_{drag} = 10^{-6}n'^2$. Plotted against time are the mean motion ratio n/n', the eccentricity e, and the resonant angle ϕ. Initial conditions are $n = 2.03n', e = 0$, and $\phi = 0$. Notice that the mean motion is decreasing as the orbit evolves away from the planet either toward the perturbing satellite (gravitational resonance) or toward synchronous orbit (Lorentz resonance). The effect of the 2:1 resonance is to change the secular reduction of the orbit's mean motion into a secular increase in its eccentricity. The resonant angle ϕ librates with small amplitude around a slightly negative value which can be found by setting equation (3) to zero and solving for ϕ.

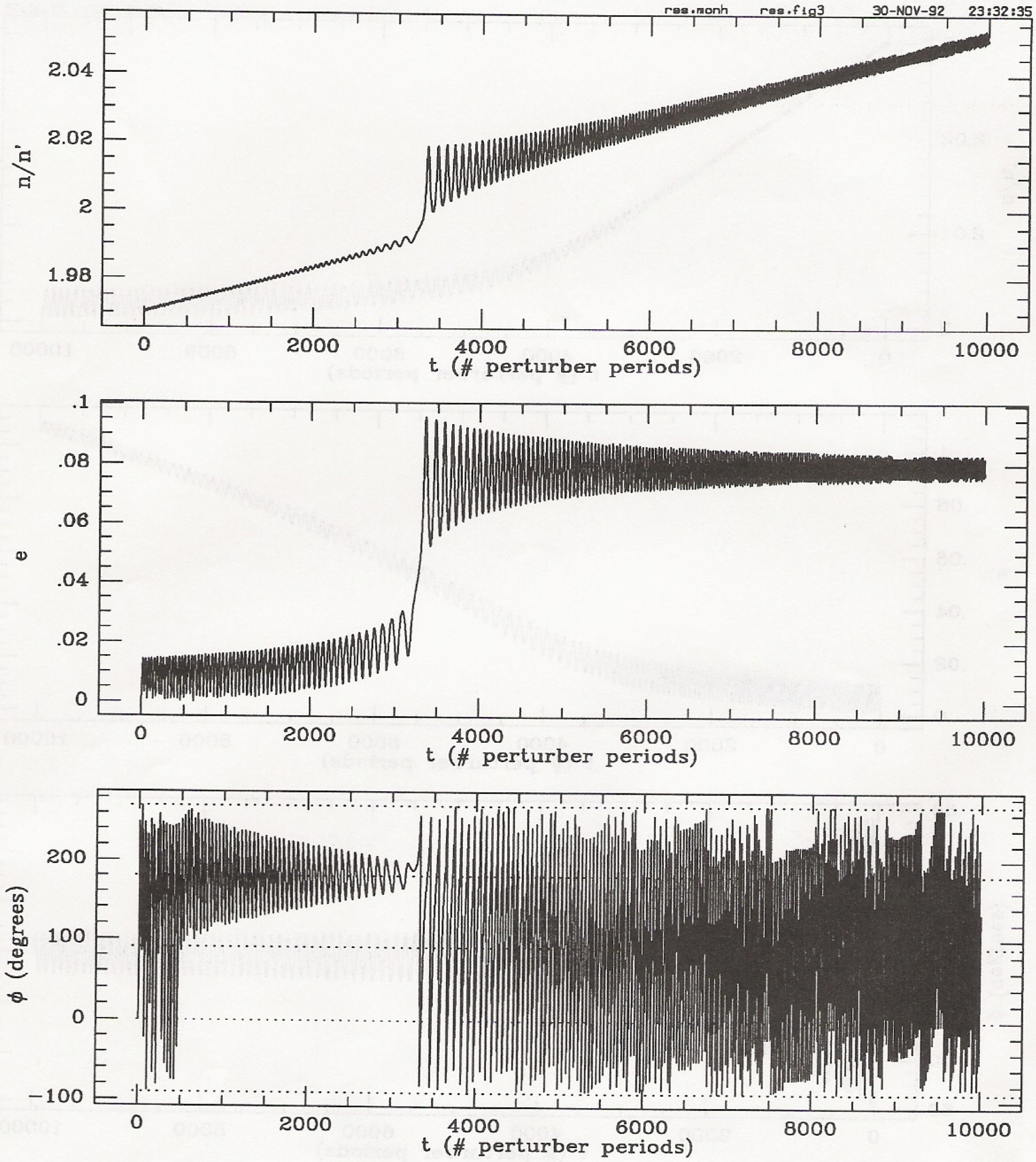

Fig. 3. <u>JUMPS AT RESONANCE</u>: A plot of the orbital evolution determined by equations (2b,c and 3) with the parameters $\beta = 10^{-4}; A_1 = A_2 = 1; \dot{n}_{drag} = -10^{-6}n'^2$. Initial conditions are $n = 1.97n', e = 0$, and $\phi = 0$. Notice that the jumps in semimajor axis and eccentricity occur simultaneously near $n \sim 2n'$ as required by equations (6) and (9). The resonant argument ϕ librates around a value near 180° until the jumps occur after which it circulates.

$$\frac{dn}{de} = \frac{3en}{A_1}, \tag{6}$$

which can be integrated to yield

$$\ln\left(\frac{n}{n_*}\right) = \frac{3e^2}{2A_2} \tag{7}$$

where n_* is an integration constant. Recalling that equation (2a-c) are accurate to only first order in eccentricity, we solve this equation to lowest order in e and find that

$$n_* = n\left[1 - \frac{3e^2}{2A_1}\right] \tag{8}$$

is a conserved constant of the motion (see /7/). Since the resonance zone is traversed quickly, equation (8) remains approximately constant during the passage. The half-width of the librating zone, $dn/2$, can be crudely estimated by setting the derivative of equation (1) equal to zero, taking $n = 2n' + dn/2$ and $\cos\phi = 1$, and solving for dn. We find $dn \approx 2n\beta A_2/e$. Inserting this into equation (6), and neglecting the difference between de and e, we find:

$$de = \left(\frac{2A_1A_2\beta}{3}\right)^{1/3}. \tag{9}$$

This case is displayed in figure 3; using the parameters from the figure caption, we calculate the jump amplitudes from equations (9) and (6) and obtain $de \approx 0.04$ and $dn \approx 0.012$ - values smaller than, but in rough agreement with, the figure.

DISCUSSION

Lorentz and gravitational resonances differ primarily in the magnitudes of the resonant strength β. For micron-sized dust grains around the jovian planets, β is orders of magnitude larger in the Lorentz case; thus Lorentz resonances are more effective at trapping dust particles and are able to induce larger orbital jumps than resonances due to a satellite's gravity. Slight additional differences arise when $A_i \neq 1$; most first-order Lorentz resonances have $A_i < 1$ which reduces the trapped growth rate (equation 5) and jump amplitude (equation 9). Despite this small difference between the two types of resonances, the equations that govern them are remarkably similar and, consequently, it is not surprising that orbital behavior at Lorentz and gravitational resonances is so alike.

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