The Mathematics of Exponential Growth

A topic that recurs frequently in this class is exponential growth. This is how population grows if there are no checks, and Darwin realized that since such growth would rapidly lead to absurd numbers, the processes that restrict such growth could favor some organisms over others and thus lead to evolution. It is also important to realize that human use of many resources is increasing exponentially, and that this cannot continue.

With that in mind, this document lays out some simple strategies for doing exponential calculations. Note that it is always possible to punch in the wrong numbers to your calculator, hence it is highly useful to have a quick way of checking your numbers approximately.

For our example, we will consider the following problem: if population grows by 1% per year, how long will it take to have 1000 times the original population? As a start, we need to understand that a growth of 1% per year means that the population increases by a *factor* of 1.01 every year. That is, if the population was 1 unit at time 0, then after one year the population would be 1.01 units, after two years the population would be $(1.01) \times (1.01) = 1.0201$ units, and so on. Therefore, you are being asked how many times 1.01 has to be multiplied by itself to get the answer 1000.

First method: guessing and adjusting.—Most calculators have a button labeled y^x, such that you can take a number y and raise it to the x power. For example, if you wanted to compute 2^5, you would hit 2 y^x 5 $=$ and (if you did it correctly) get the answer 32.

How does this help? Let's suppose that you have absolutely no idea how many times you need to multiply 1.01 by itself to get 1000. Then you can give it a try with some numbers. If you guess too low, you raise the number of multiplications; if you guess too high, you lower the number. Say that you guess 10 times. $1.01^{10} = 1.105$ (approximately), so obviously you need to guess higher. $1.01^{100} = 2.7$, so you need higher still. $1.01^{1000} = 20959$, so that's too high. $1.01^{500} = 144.8$, so that's too low. $1.01^{750} = 1741.9$. $1.01^{700} = 1059$, so that's pretty close. Notice that doubling the exponent does *not* double the number; in fact, it squares the number. That means that small variations in your exponent can make a big difference. Since 700 is pretty close to the right value, you shouldn't adjust it much. Experimentation shows that $1.01^{694} = 997.8$ is the closest whole number guess. Therefore, 694 years is about right. Note that you can get to this number with just a few calculations.

Second method: the law of 70, and $2^{10} \approx 10^3$.—A useful rule is that if the growth rate is x percent in some time period, then you need roughly $70/x$ time periods to double the quantity (we assume here that x is much less than 100). For our example, where the growth rate is 1% per year, that suggests about 70 years for a doubling. The next point is that because $2^{10} = 1024 \approx 1000 = 10^3$, if you need to know how many doublings are required, take the power of 10 and multiply it by $(10/3)$. For our case, we need a factor of $1000 = 10^3$,

so we need about 10 doublings. Each doubling takes about 70 years, so a rough estimate is about 700 years. As we just saw, this is very close to correct.

Third method: logarithms.—This is the way to get the exact answer. The particular problem we have to solve is $1.01^y = 1000$, where we need to determine y. If you need a refresher on logarithms, just Google "logarithm". If you take the logarithm of both sides you get

$$\begin{aligned}
1.01^y &= 1000 \\
y\log(1.01) &= \log(1000) \\
y &= \log(1000)/\log(1.01) = 694.224...\ .
\end{aligned} \tag{1}$$

Note that it does not matter what base you use for the logarithm; it could be base 10, it could be a natural logarithm with base $e = 2.71828...$, it could be anything.

Practice problems

1. Suppose you have a 5% growth rate every year. How long would it take to increase the quantity by a factor of a million?

2. Suppose you have a 0.3% growth rate every year. How long would it take to increase the quantity by a factor of 20,000?

3. Suppose you have a 2% growth rate every five years. How long would it take to increase the quantity by a factor of 100?

4. (to test understanding!) Suppose you have a *negative* 1% change every year. How long would it take to *decrease* the population by a factor of 2? Note that this means you are solving $0.99^y = 0.5$. *Hint:* take the reciprocal of both sides.