# 1. Isotropic Scattering in Slabs: Comparison of Methods

One use of the spline-based matrix transforms given here is to provide accurate solutions which may serve as test cases for more general methods. In particular, we would like to test Monte Carlo codes which can then be applied to cases which do not have plane-parallel geometry. The simplest such tests involve isotropic scattering (without polarization) in plane parallel slabs of finite optical thickness. Let us start with the case of sources of radiation distributed uniformly throughout slab. After the radiation is emitted, let it scatter without absorption until it escapes. We want to find the resultant source function and the angular dependence of the emergent radiation. The equation for the source function $S$ in this case is

$$S_\nu(\tau_\nu) \;=\; \Lambda_{\tau_\nu}(S_\nu) \;+\; S^*_\nu(\tau_\nu) \quad , \tag{1}$$

where $S^*$ is the uniformly distributed source term. This represents a process that emits without absorbing – and it also corresponds to the limiting case of situations for which $\lambda_\nu << 1$, so that $(1-\lambda_\nu)$ approaches unity. The matrix equation then is given by

$$[I_{ij} \;-\; \Lambda_{ij}]\; S_i \;=\; S^*_i \tag{2}$$

with the solution

$$S_i \;=\; [I_{ij} \;-\; \Lambda_{ij}]^{-1}\; S^*_i \tag{3}$$

The J routine in the file **Uniform.ijs** provides this solution and the resulting emergent intensity as a function of $\mu = \cos(\theta)$.

To set up the Monte Carlo approach, let us consider an x-y-z coordinate system with its origin on the mid-plane of the slab and the z-axis perpendicular to the slab. Then we see that the only variables of interest are the z coordinate of the scattering and the angle $\mu = \cos(\theta)$ of the photons path with respect to the z-axis. The x and y coordinates are not needed – we do not care at what point the photons emerge or with what $\phi$ angle, only with what $\theta$.

We set up a collection of photons uniformly distributed in z. We next assign a random $\theta$ direction for each – this is given by assigning each a $\mu$ sampled over the interval [-1,1]. For a photon at z traveling in direction $\mu$, the distance to the edge of the slab (and escape) is given by

$$\tau_{esc} \;=\; \frac{(T/2)}{|\mu|} \;-\; \frac{z}{\mu}\;, \qquad \text{for} \;\; (T/2) \geq z \geq (-T/2) \quad \text{and} \quad 1 \geq \mu \geq -1 \;\;, \tag{4}$$

and thus the fraction that would escape along that path is $f_{esc} = e^{-\tau_{esc}}$, while a fraction $(1 - f_{esc}) = 1 - e^{-\tau_{esc}}$ will not escape but will be scattered somewhere along the path before reaching the edge of the slab.

A uniform sample of the optical distances traveled by photons in an infinite medium is given by $\tau = -\ln(1-r)$, where $r$ is a number drawn at random from the interval [0,1]. (E.g., $r$ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 leads to $\tau$ = 0.105, 0.223, 0.357, 0.511, 0.693, 0.916, 1.204, 1.609, 2.303)

Now the most direct approach is to choose an $r$ for each photon, and if the corresponding $\tau$ is $\tau \geq \tau_{esc}$, the photon escapes, while if $\tau < \tau_{esc}$, we keep the photon, letting it scatter into a new random $\mu$ at location $z_{new} = z_{old} + \mu\ \tau$ . Then when all, or nearly all, of the photons have escaped we are done. Starting with a large group of photons, each scattering step will process a lesser number of photons than that before, and this requires some book-keeping.

To demonstrate the simplest code, we will instead adopt *photon splitting*. Each "photon" (suppose they represent a bundle of photons) has its own intensity $I_{old}$. Upon scattering, a fraction $f_{esc}I_{old}$ of the photon's intensity is recorded as escaped, while the remaining fraction $I_{new} = (1 - f_{esc})I_{old}$ becomes a new (less intense) photon which is scattered in the slab. The location of the scattering of the $I_{new}$ photon is $\tau_{sc} = -\ln(1-(1-f_{esc})r)$, where $r$ is uniform over [0,1]. We see that $0 \leq \tau_{sc} < \tau_{esc}$ . The new $z$ then becomes $z_{new} = z_{old} + \mu\ \tau_{sc}$ . We continue this process, scattering the same number of photons each time until the remaining intensities are sufficiently small.

In the J implementation, each escape/scattering is done by a verb "Step", so that we can use the "power" function ^ : to do nscat scatterings: Step^ :(nscat). We also need a routine "Sort" to accumulate the escaping photons into a set of intervals in $\mu$. The whole routine is given in **Monte_Uslab.ijs** and is quite compact.

A similar problem is that of a scattering slab with the source of photons located at the mid-plane. The change in the Monte Carlo code is trivial: we just replace the initial assignment of random $z$ values with $z = 0$ for all photons. The matrix solution is not so simple, as we do not want to introduce a delta function for $S^*$. We can mitigate this somewhat by letting the source term $S^*(\tau)$ represent the first scattering of photons emitted from the mid-plane. Now if radiation is emitted isotropically from the mid-plane with intensity $I_0$, the intensity reaching a layer at $\tau$ along direction $\mu$ will be $I_0\ e^{-\tau/|\mu|}$. Then we see that the radiation scattered from this beam in the layer $\tau$ to $\tau + d\tau$ must be $I_0\ e^{-\tau/|\mu|}\ d\tau/|\mu|$. Then the total radiation scattered for the first time in this layer (let's assume $\mu > 0$, the upper half plane) is given by

$$S^*(\tau)d\tau \ = \ \int_0^1 I_0\ e^{-\tau/\mu}\ \frac{d\tau}{\mu}\ d\mu \ = \ I_0\ \left\{\int_0^1 e^{-\tau/\mu}\ \frac{d\mu}{\mu}\right\} d\tau \ = \ I_0\ E_1(\tau)d\tau \qquad (5)$$

This may not seem to solve our problem, as $S^*(\tau) = I_0\ E_1(\tau)$ is still singular at $\tau = 0$, the mid-plane, but it is in fact better as it is just a *logarithmic* singularity. If we lay down a grid that is finely spaced near the mid-plane as well as near the surface, and evaluate the last $E_1(\tau)$ not at $\tau = 0$, but at a point midway between the last two $\tau$'s, we will make little error. Note that if we wish the total (physical) flux ($2\pi \int_0^1 I(\mu)\mu d\mu$) emerging from both faces of the slab to be unity per unit area ( 1/2 from each face), we must normalize the intensity $I_0$ properly: $I_0 = 1/8\pi$. Also, note that some flux emerges directly with out being scattered, and this is just $E_2(T/2)$, where $T/2$ is the half-thickness of the slab.

The J code for this approach is given in the file **Mid_plane.ijs**, while the Monte Carlo code is in **MC_mid-slab.ijs**.

Another problem of interest is a slab illuminated by an external beam incident at some angle $\theta_i$, corresponding to a $\cos(\theta)$ of $\mu_i$. In this case, the source term at a level $\tau$ is the energy from the incoming beam which is first scattered in that layer. This is just

$$S^*(\tau) \;=\; \frac{1}{4\pi\ \mu_i}\ e^{-\tau/\mu_i} \quad , \tag{6}$$

where the normalization involves $\mu_i^{-1}$ so that a beam of unit intensity will produce a total flux (from the top plus the bottom) *per unit surface area* of unity. (Note that a beam incident at a shallow angle will be spread out over a large area of the slab's surface.) The $1/(4\pi)$ is, again, because the source term wants to be per steradian (the emission per unit volume is $4\pi S^*$).

The matrix code to solve this in J is given in **Beam.ijs**. To solve the problem by a Monte Carlo method, we set the initial intensity of the photons to $I_0 = (1 - e^{-T/\mu_i})$, where $T$ is the optical thickness of the slab normal to its surface, because a fraction $e^{-T/\mu_i}$ of the beam passes through the slab without scattering. We then scatter all these photons with a distribution in $z$ based on equation (6). This turns out to be

$$z \;=\; \frac{1}{2}T \;-\; \mu_i\tau_{sc} \quad ,\text{where} \qquad \tau_{sc} \;=\; -\ln\left[1 \;-\; (1 - e^{-T/\mu_i})r\right] \quad , \tag{7}$$

and, as before, $r$ is a random number drawn from the interval [0,1]. The J code for this scheme is given in **MC_beam.ijs**. Fig. 1 shows a comparison of the "exact" results with a Monte Carlo run of $4 \times 10^6$ photons scattered 30 times. The results are close except for one point in the $\mu = 0.05$ bin – the intensity at small $\mu$ has the largest error, as the intensity has been scaled up by $1/\mu$ from a relatively small number of escapes at shallow angles.

Finally, it should be clear that both the matrix and Monte Carlo codes can be generalized to treat arbitrary variations of the source $S^*(\tau)$ or an arbitrary variation of external illumination with angle. Also, if there is true absorption present, we can see that the equation will be

$$S_\nu(\tau_\nu) \;=\; [1 - \lambda_\nu(\tau_\nu)]\Lambda_{\tau_\nu}(S_\nu) \;+\; \lambda_\nu(\tau_\nu)\ B_\nu(\tau_\nu) \;+\; S^*_\nu(\tau_\nu) \quad , \tag{8}$$

and the corresponding solution for the source function is

$$S_i \;=\; \{I_{ij} \;-\; (1 - \lambda_i)\Lambda_{ij}\}^{-1}\ [\lambda_i B_i \;+\; S^*_i] \tag{9}$$

The presence of absorption ($\lambda \neq 0$) will speed the convergence of the Monte Carlo solutions, since in addition to escapes, the scattered intensity will decrease as $(1 - \lambda)^{nscat}$.

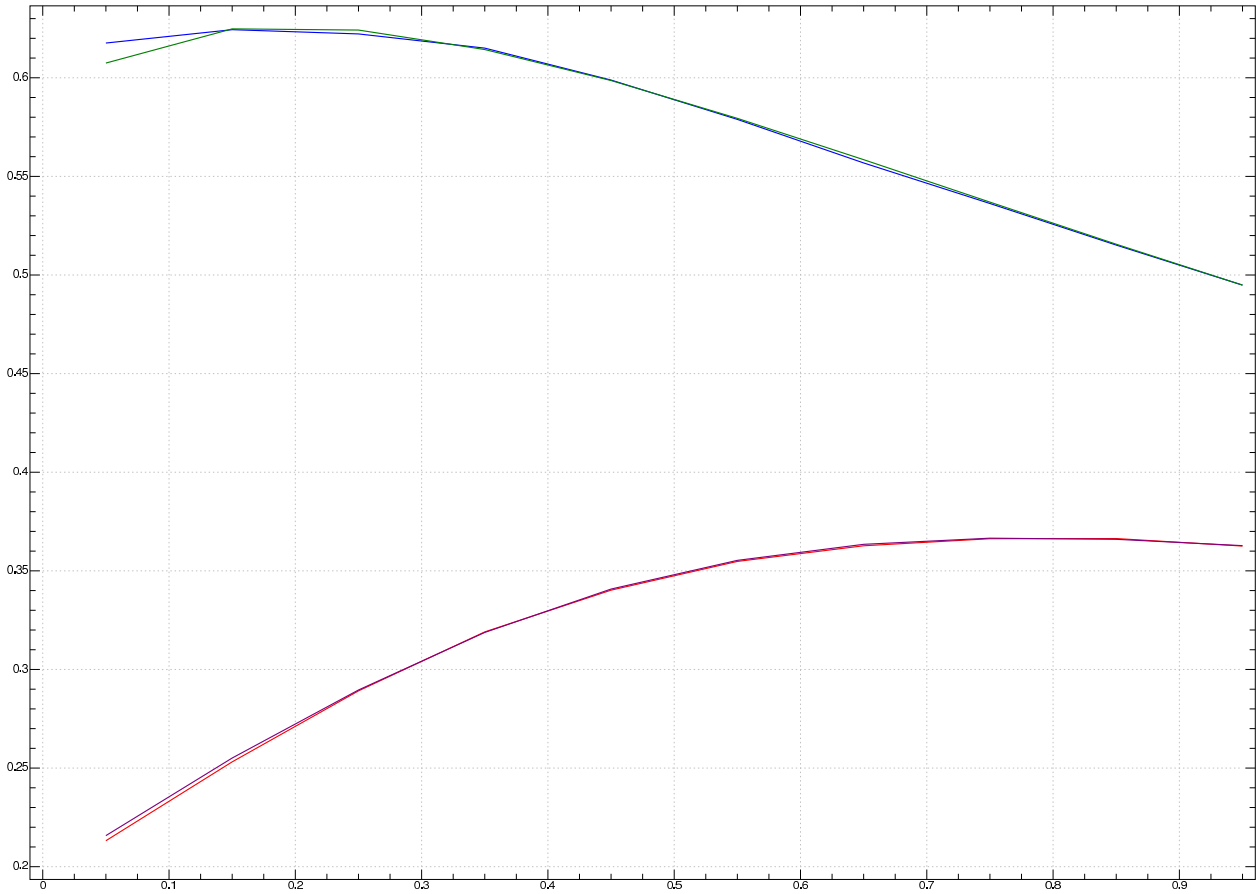


Fig. 1.— A comparison of *Beam.ijs* with *MC_beam.ijs* for a beam entering at $\mu_i = 0.864665$ ($\theta = 30^o$) into slab of total optical depth $T = 2$. The plot is (emergent intensity) vs. $\mu$.
From top of slab: blue=Monte Carlo; green=matrix transform
From bottom of slab: red=Monte Carlo; purple=matrix transform