Tutorial on Gravitational Radiation

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ,ϕ) is $A(r,\theta,\phi)$, the flux through a spherical surface at r is $F(r,\theta,\phi) \propto A^2(r,\theta,\phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r,\theta,\phi) = A(r)$, this means that the luminosity at a distance r is $L(r) \propto A^2(r)4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r) \propto 1/r^2$, implying that $L(r) \propto 1/r^2$ and hence no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

For gravitational radiation, what can vary? Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r})d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r})\mathbf{r}d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The equivalent of the magnetic dipolar moment is $\int \rho(\mathbf{r})\mathbf{r} \times \mathbf{v}(\mathbf{r})d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r})r_i r_j d^3r$. This is not conserved, therefore there can be quadrupolar gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, hence it does not produce radiation. No matter how violent an explosion or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains axisymmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (i.e., individually unresolved sources with random phases; the most interesting of these would be a background of gravitational waves from the early universe).

What is the approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source? We argued that the lowest order radiation had to be quadrupolar, and hence depend on the quadrupole moment I. This moment is $I_{ij} = \int \rho r_i r_j d^3x$, so it has dimensions MR^2, where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r\ . \tag{1}$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective "units", but we can't, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can't emit anything. Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r}\frac{\partial^2 (MR^2)}{\partial t^2}\ . \tag{2}$$

Now what? To get back to physical units we have to restore factors of G and c. It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2 (MR^2)}{\partial t^2}\ . \tag{3}$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless M and R are large, the system is changing fast, and r is small, the metric perturbation is minuscule.

Let's make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2 / M$, the semimajor axis is a, and the orbital frequency Ω is therefore given by $\Omega^2 a^3 = M$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G/c^4)(1/r)(\mu M/a)\ . \tag{4}$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal mass system

$$h \approx 10^{-22} \left(\frac{M}{2.8\,M_\odot}\right)^{5/3} \left(\frac{0.01 \text{ sec}}{P}\right)^{2/3} \left(\frac{100 \text{ Mpc}}{r}\right) , \tag{5}$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That's why it is so challenging to detect these systems!

Remarkably, though, the flux of energy is *not* tiny. To see this, let's calculate the flux given some dimensionless amplitude h. The flux has to be proportional to the square of the amplitude and also the square of the frequency f: $F \sim h^2 f^2$. This currently has units of time squared, but the physical units of flux are energy per time per area. Replacing factors of G and c, we find that the flux is

$$F \sim (c^3/G)h^2 f^2 \ . \tag{6}$$

Now the prefactor is *enormous*! For the double neutron star system above, with $h \sim 10^{-22}$ and $f \sim 100$ Hz, this gives a flux of a few hundredths of an erg cm^{-2} s^{-1}. For comparison, the flux from Sirius, the brightest star in the night sky, is about 10^{-4} erg cm^{-2} s^{-1}! That means that if you could somehow absorb gravitational radiation perfectly with your eyes, you would find untold billions of sources brighter than every star except the Sun. What this really implies, of course, is that gravitational radiation interacts *very* weakly with matter, which again means that it is mighty challenging to detect.

Let's get an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, the density, we can say that the maximum frequency involving an object of density ρ is $f_{\rm max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave through the object can't be greater than $\sim (G\rho)^{1/2}$. Therefore, this is a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-3} - 10^{-6}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1 - 10$ Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000 - 2000$ Hz, and that for black holes the limit depends inversely on mass (and also spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is on the order of $10^4(M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at lower frequencies (more on that later).

Now suppose that the binary is well-separated, so that the components can be treated as points and we only need take the lowest order contributions to gravitational radiation. Temporarily restricting our attention to circular binaries, how will their frequency and amplitude evolve with time?

Let the masses be m_1 and m_2, and the orbital separation be R. We argued in the previous lecture that the amplitude a distance $r \gg R$ from this source is $h \sim (\mu/r)(M/R)$, where $M \equiv m_1+m_2$ is the total mass and $\mu \equiv m_1 m_2/M$ is the reduced mass. We can rewrite the amplitude using $f \sim (M/R^3)^{1/2}$, to read

$$\begin{aligned} h &\sim \mu M^{2/3} f^{2/3}/r \\ &\sim M_{ch}^{5/3} f^{2/3}/r \end{aligned} \tag{7}$$

where M_{ch} is the "chirp mass", defined by $M_{ch}^{5/3} = \mu M^{2/3}$. The chirp mass is named that because it is this combination of μ and M that determines how fast the binary sweeps, or chirps, through a frequency band. When the constants are put in, the dimensionless gravitational wave strain amplitude (i.e., the fractional amount by which a separation changes as a wave goes by) measured a distance r from a circular binary of masses M and m with a binary orbital frequency $f_{\rm bin}$ is (Schutz 1997)

$$h = 2(4\pi)^{1/3}\frac{G^{5/3}}{c^4} f_{\rm GW}^{2/3} M_{ch}^{5/3}\frac{1}{r}\ , \tag{8}$$

where $f_{\rm GW}$ is the gravitational wave frequency. Redshifts have not been included in this formula.

The luminosity in gravitational radiation is then

$$\begin{aligned} L &\sim 4\pi r^2 f^2 h^2 \\ &\sim M_{ch}^{10/3} f^{10/3} \\ &\sim \mu^2 M^3/R^5\ . \end{aligned} \tag{9}$$

The total energy of a circular binary of radius R is $E_{\rm tot} = -G\mu M/(2R)$, so we have

$$\begin{aligned} dE/dt &\sim \mu^2 M^3/R^5 \\ \mu M/(2R^2)(dR/dt) &\sim \mu^2 M^3/R^5 \\ dR/dt &\sim \mu M^2/R^3 \ . \end{aligned} \tag{10}$$

What if the binary orbit is eccentric? The formulae are then more complicated, because one must then average properly over the orbit. This was done first to lowest order by Peters and Matthews (1963) and Peters (1964), by calculating the energy and angular momentum radiated at lowest (quadrupolar) order, and determining the change in orbital elements that would occur if the binary completed a full Keplerian ellipse in its orbit. That is, they assumed that to lowest order, they could have the binary move as if it experienced only Newtonian gravity, and integrate the losses along that path.

Before quoting the results, we can understand one qualitative aspect of the radiation when the orbits are elliptical. From our derivation for circular orbits, we see that the radiation is emitted much more strongly when the separation is small, because $L \sim R^{-5}$. Consider what this would mean for a very eccentric orbit $(1-e) \ll 1$. Most of the radiation would be emitted at pericenter, hence this would have the character of an impulsive force. With such a force, the orbit will return to where the impulse was imparted. That means that the pericenter distance would remain roughly constant, while the energy losses decreased the apocenter distance. As a consequence, the eccentricity decreases. In general, gravitational radiation will decrease the eccentricity of an orbit.

Sample problems:

1. What is the number of sources in a given frequency bin for a steady state population of circular sources?

Answer:

For a circular orbit, de/dt is irrelevant but the change in semimajor axis is

$$\langle \frac{da}{dt} \rangle = -\frac{64}{5}\frac{G^3\mu M^2}{c^5 a^3} \ . \tag{11}$$

Remembering that $f_{\rm GW} = (1/\pi)(GM/a^3)^{1/2}$, that means that

$$da/dt = -\frac{2}{3}\left(\frac{GM}{\pi^2}\right)^{1/3} f^{-5/3}(df/dt) = -(64/5)\frac{G^3\mu M^2}{c^5 a^3} \ . \tag{12}$$

From this, we get

$$\frac{d\ln f}{dt} = \frac{96}{5}\left(\frac{GM}{\pi^2}\right)^{-1/3}\frac{G^2\mu M}{c^5}\pi^2 f^{8/3} \ . \tag{13}$$

The reciprocal of this gives the characteristic time $T_{\rm insp}$. Therefore,

$$f_{\rm min} = \left[\frac{96}{5}\left(\frac{GM}{\pi^2}\right)^{-1/3}\frac{G^2\mu M}{c^5}\pi^2 T_{\rm insp}\right]^{-3/8} \ . \tag{14}$$

For two white dwarfs of mass $0.6\,M_\odot$, $M = 1.2\,M_\odot$ and $\mu = 0.3\,M_\odot$. Inserting $T_{\rm insp} = 10^{10}$ yr $\approx 3\times10^{17}$ s gives $f_{\rm min} \approx 9\times10^{-5}$ Hz.

2. Below $f_{\rm min}$ the distribution dN/df of sources with frequency will depend on their birth population. Above it, gravitational radiation controls the distribution. Derive the dependence of dN/df on f for $f > f_{\rm min}$ (the normalization is not important).

Answer:
The number in a frequency bin between f and $f + df$ (with df fixed) is just proportional to the time spent in that bin. We found that $d\ln f/dt \propto f^{8/3}$, meaning that $df/dt \propto f^{11/3}$, so $dt \propto f^{-11/3}df$. Therefore, $dN/df \propto f^{-11/3}$. Pretty steep dependence!

3. Suppose there are 10^9 WD-WD binaries at frequencies $f_{\rm min} < f < 0.1$ Hz. To within a factor of 2, compute the frequency $f_{\rm res}$ above which you expect an average of less than one WD-WD binary per $df = 10^{-8}$ Hz frequency bin (this is 1/(3 yr), or about the frequency resolution expected for the LISA experiment). Very roughly speaking, above $f_{\rm res}$ one can identify individual WD-WD binaries, whereas below it is the confusion limit.

Answer:
If there are $N_{\rm tot}$ total binaries above some frequency $f_{\rm min}$, then

$$\int_{f_{\rm min}}^{\infty} A f^{-11/3} df = N_{\rm tot} \ , \tag{15}$$

where A is a normalization constant we need to determine. Note that, strictly, the integral would actually have an upper limit of 0.1 Hz instead of infinity, but as the integral is dominated by the lower limit it is easier to go to ∞.

Then $A = (8/3) f_{\rm min}^{8/3} N_{\rm tot}$, and the number between f and $f + df$ is $N(f) = A f^{-11/3} df$. Putting in $f_{\rm min} = 9 \times 10^{-5}$ Hz, $N_{\rm tot} = 10^9$, and $df = 10^{-8}$ Hz, we solve for $N(f) = 1$ to find $f_{\rm res} \approx 3 \times 10^{-3}$ Hz. By the way, it's not as if no bins above this frequency will have multiple WD-WD sources; plenty will, it just gets rarer for higher frequencies.