# Evidence for eccentricity in the population of binary black holes observed by LIGO-Virgo-KAGRA

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Binary black holes (BBHs) in eccentric orbits produce distinct modulations in the emitted gravitational waves (GWs). The measurement of orbital eccentricity can provide robust evidence for dynamical binary formation channels. We analyze 57 GW events from the first, second and third observing runs of the LIGO-Virgo-KAGRA (LVK) Collaboration using a multipolar aligned-spin inspiral-merger-ringdown waveform model with two eccentric parameters: eccentricity and relativistic anomaly. This is made computationally feasible with the machine-learning code `DINGO`, which accelerates inference by 2-3 orders of magnitude compared to traditional inference techniques. First, when using a uniform prior on the eccentricity, we find eccentric aligned-spin against quasi-circular aligned-spin $\log_{10}$ Bayes factors of 1.84 to 4.75 (depending on the glitch mitigation) for GW200129, 3.0 for GW190701 and 1.77 for GW200208_22. We measure $e_{\rm gw,\,10Hz}$ to be $0.27^{+0.10}_{-0.12}$ to $0.17^{+0.14}_{-0.13}$ for GW200129, $0.35^{+0.32}_{-0.11}$ for GW190701 and $0.35^{+0.18}_{-0.21}$ for GW200208_22. Second, we find $\log_{10}$ Bayes factors between the eccentric aligned-spin versus quasi-circular precessing-spin hypothesis between 1.43 and 4.92 for GW200129, 2.61 for GW190701 and 1.23 for GW200208_22. Third, our analysis does not show evidence for eccentricity in GW190521, which has an eccentric aligned-spin against quasi-circular aligned-spin $\log_{10}$ Bayes factor of 0.04. Fourth, we estimate that if we neglect the spin-precession and use an astrophysical prior on the rate of eccentric BBHs, the probability of one out of the 57 events being eccentric is greater than 99.5% or $(100-8.4\times10^{-4})$% (depending on the glitch mitigation). Fifth, we study the impact on parameter estimation when neglecting either eccentricity in quasi-circular models or higher modes in eccentric models for GW events. These results underscore the importance of including eccentric parameters in the characterization of BBHs for the upcoming observing runs of the LVK Collaboration and for future detectors on the ground and in space, which will probe a more diverse BBH population.

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## I. INTRODUCTION

Gravitational waves (GWs) have been a gold mine for astrophysical characterization and discovery since the first detection in 2015 [1–12]. They are a clean probe for binary black holes (BBHs) due to their weak interaction with matter. GWs have been used to infer the masses, spins [13–19] eccentric parameters [20–24], and test General Relativity (GR) [25–30]. BBH properties are important as they can give insight into the binary's formation channels [31–36].

While stellar-mass black holes (BHs) form from the collapse of stars with masses $\gtrsim 20M_\odot$ [37–39], and primordial BHs may form from over-dense regions in the early Universe [40], the formation of a BBH is not trivial. One cannot assume that all stellar binaries will become BBHs. If the initial separation of two $\sim 10M_\odot$ compact objects in a quasi-circular orbit is greater than $\sim 20R_\odot$, then they will not merge in a Hubble time ($\sim 14$ Gyr) from gravitational radiation alone [41]. However, if the initial separation of this binary is less than $\sim 20R_\odot$, the stars merge before forming BHs (see, e.g., Figs. 1 and 2 in Ref. [42]).

Thus, to explain the formation of BBHs, two classes of formation mechanisms of BBHs have been proposed in the literature: isolated binary evolution and dynamical formation. Within these categories, the former has sub-channels including common-envelope evolution [39, 43–49], chemically-homogenous evolution [50, 51] and Population III stars [52]. Similarly, the dynamical channel has sub-channels, e.g., binaries arising from triple interactions [53–57], von Zeipel-Kozai-Lidov (ZKL) oscillations [58–63], binary-binary interactions [64], and binary-single interactions [65, 66]. These possibilities had been proposed in Refs. [67–74], and further investigated in Refs. [75–82]

It is unclear what fraction of BBHs arise from each formation channel. Multiple channels may operate simultaneously making the origin of BBHs even harder to pinpoint. But measuring eccentricity in a BBH can serve as a reliable indicator of a dynamical formation channel [63]. This is because isolated binaries that are initially eccentric would circularize by the time they enter the LIGO-Virgo-KAGRA (LVK) band [41]. So by studying the effect of eccentricity, we can help discriminate what fraction of BBHs originate from dynamical interactions or IB evolution [31].

There is also good reason to believe we may observe a few eccentric events in the LVK observing runs [7, 11, 12] or future GW observatories, such as the Einstein Telescope [83], Cosmic Explorer [84] and LISA [85]. For instance, there have been robust simulations performed to determine the expected number of BBHs from the dynamical-formation channel. Simulations which include the post-Newtonian (PN) description of the orbital dynamics in globular star clusters show

that ≈ 5% of dynamical mergers have eccentricities greater than 0.1 This estimate refers to an astrophysical definition of eccentricity [63, 86], not the quasi-Keplerian definition. at 10 Hz (in the detector frame) [65, 87, 88]. Estimates from galactic nuclei predict higher rates of eccentricity with up to 70% of mergers predicted to have eccentricities larger than 0.1[1] at 10 Hz [89–91]. These high eccentricity mergers could be detected by upcoming runs of the LVK collaboration. It is then important to have an eccentric waveform model to analyze the data.

Taking eccentricity into account is also important as ignoring it can lead to systematic biases in parameter estimation [24, 92–103]. So far, eccentric waveform models have not been used for analysis of the LVK GW transient catalogues (GWTC) [1–4]. This limitation may cause biases and have implications on hierarchical BBH population studies and tests of GR [104–109].

Several parameter-estimation studies that included one eccentric parameter in the waveform model have been carried out in the last few years [20–23, 95, 97, 100, 110–112]. There have also been parameter estimation studies that use two parameters to explore the effect of eccentricity [24, 94, 113]. In particular, Ref. [94] performed inference by fixing the binary at apastron and sampling on the initial eccentricity and the average starting frequency between apastron and periastron. On the other hand, Ref. [24] fixes the orbit averaged starting frequency and samples directly on the relativistic anomaly and the eccentricity. We note that sampling on starting frequency and eccentricity at fixed relativistic anomaly may not be equivalent to sampling on eccentricity and relativistic anomaly at a fixed starting frequency. In principle these methods can probe the same parameter space. However, if sampling on the starting frequency, one has to choose arbitrary prior bounds on the starting frequency. If there are templates with a better match to the GW outside of these bounds, this method does not cover the same parameter space as the method which fixes the starting frequency and samples on the relativistic anomaly and eccentricity. In addition, sampling on the starting frequency will change the reference point where the binary properties, such as the reference phase, are measured. Finally, Ref. [113] samples on the energy and angular momentum of the binary at a fixed instantaneous starting frequency. We note that whereas Ref. [24] deals with (bound) eccentric orbits, Ref. [113] focuses on hyperbolic orbits.

Due to the computational cost, the aforementioned analyses were restricted to either sampling on non-eccentric parameters and reweighting with eccentric models, or to a small number of GW events when using Bayesian methods that sample directly on the eccentric parameters. Here, we employ the Deep INference for Gravitational-wave Observations (`DINGO`) code to sample a large number of GW events. As in Refs. [24, 94], we perform inference on the two eccentric parameters: eccentricity and relativistic anomaly. This is important, as ignoring the relativistic anomaly can lead to biases in parameter estimation [24, 114]. `DINGO` has already been used to analyze a subset of the spin-precessing BBHs in the first and third observing runs [115]. Remarkably, `DINGO` yields posterior distributions within $\mathcal{O}$(min - hours) without sacrificing accuracy [115, 116]. However, it is currently limited to GW signals shorter than 16 sec.

In this study, we employ a set of waveform models from the effective-one-body (EOB) family `SEOBNR`. We use the multipolar aligned-spin quasi-circular model `SEOBNRv4HM` [117, 118] and the multipolar aligned-spin eccentric model `SEOBNRv4EHM` [119, 120] to analyze 57 GW events from the first (O1), second (O2) and third (O3) observing runs of the LVK Collaboration [4, 121, 122]. We find $\log_{10}$ Bayes factors, between the eccentric against quasi-circular hypothesis, greater than 1 for three events. We also do a comprehensive study to understand the impact of glitch subtraction for one of the GW events (GW200129), which happens to be the BBH with the largest evidence for eccentricity. Confirming the recent results in Ref. [24] we do not find evidence for eccentricity in GW190521, which previous analyses had suggested to be eccentric [21, 110].

It has been suggested that for some regions of the parameter space, and at certain stages of the coalescence, spin-precession can be mistaken for eccentricity in parameter-estimation studies [123, 124]. Thus, for comparison, we also employ two multipolar spin-precessing waveform models for quasi-circular orbits. One is from the EOB waveform family, `SEOBNRv4PHM` [125–128], and the other is from the numerical-relativity surrogate family, `NRSur7dq4` [129]. Unfortunately, inspiral-merger-ringdown waveform models with spin-precession and eccentricity are not yet available, with the exception of the recent Ref. [130], whose code is not available to us. We also explore the impact of ignoring eccentricity and higher harmonics in parameter estimation.

When trying to determine if a BBH is eccentric, we must also consider the relative merger rates of eccentric and quasi-circular BBHs. This can be done by incorporating astrophysical knowledge from N-body simulations or semi-analytic calculations. We can also directly infer the rates from the GW data. In this paper, we combine both approaches to first generate a prior on the rate of eccentric and quasi-circular BBHs based on results from the literature, and then incorporate GW information. This results in a new way to quantify the probability that a GW comes from an eccentric BBH, similar to the $p_{\rm astro}$ reported in LVK analyses [131]. This approach also allows us to quantify the probability of eccentricity in the population of BBHs.

The paper is organized as follows. In Sec. II, we give an overview of the Bayesian inference techniques used in this study. In particular, in Sec. II A we discuss Bayes theorem and how to compute odds ratios, in Sec. II B we compute the probability a GW signal is eccentric based on the ensemble of events, and in Sec. II C we review the machine-learning code `DINGO`. In Sec. III, we introduce the waveform models used in this paper, and review the method introduced in Refs. [132, 133] to measure eccentricity directly from the waveform. Furthermore, in Sec. IV A we discuss the settings of the trained networks and priors employed in the `DINGO` analysis, while in Sec. IV B we validate the neural-networks with zero-noise synthetic signals and comparisons to parallel `Bilby` (`pBilby`) [134]. From Secs. IV C through IV F, we present the analysis of 57 GWs and discuss in depth the

events that support eccentricity. In Secs. V A and V B, we investigate the impact on parameter estimation of neglecting eccentricity and higher modes in waveform models, respectively. In Sec. VI A, we incorporate astrophysical BBH merger-rate estimates of eccentric and quasi-circular events into our analysis, while in Sec. VI B, we estimate the probability of GW signals being eccentric and the probability that eccentricity exists in the population of GWs analyzed. In Sec. VII, we summarize our main conclusions, comparing our results with the literature and suggesting future work. Finally, in Appendix A, we perform analyses of GW200129 using either only the LIGO-Hanford or the LIGO-Livingston detector. In Appendix B, we describe a method to compute the probability there exists an eccentric event in the ensemble of GWs analyzed. In Appendix C, we present the kick velocity posterior distribution of GW200129 when using the multipolar aligned-spin eccentric model `SEOBNRv4EHM`, and compare it to results using the multipolar spin-precessing quasi-circualr model `NRSur7dq4`. Lastly, in Appendix D, we present the results of synthetic-signal injections aimed at determining if eccentricity could be mistaken for spin-precession in GW200129.

## NOTATION

In this paper we use natural units, and set $G = c = 1$ unless otherwise specified. Individual component masses are denoted $m_1, m_2$, while the total mass is $M \equiv m_1 + m_2$, and the chirp mass is $\mathcal{M} = (m_1 m_2)^{3/5} M^{-1/5}$. We use the convention, $q \equiv m_2/m_1 \leq 1$, and define $\mu \equiv m_1 m_2/M$ and $\nu \equiv \mu/M$. We quote the masses in both the source and detector frame denoting them as $M_{\rm src} \equiv M$ and $M_{\rm det} \equiv M(1+z)$, respectively. Here $z$ is the redshift of the source.

Although we mostly consider BHs with spins $\boldsymbol{S}_{1,2}$ aligned (or anti-aligned) with the directiom perpendicular to the orbital plane, we also carry out a few studies with spin-precessing BHs. Thus, generically, we introduce the dimensionless spin

$$\chi_i \equiv \frac{\boldsymbol{S_i}\cdot\hat{\boldsymbol{L}}}{m_i^2} \quad i = 1,2\,, \tag{1.1}$$

which takes values in the range $[-1,+1]$, where $\hat{\boldsymbol{L}}$ is the unit vector pointing in the direction perpendicular to the instantaneous orbital plane. In parameter estimation one often uses $\chi_{\rm eff}$ as it is better constrained [135–138]. This is defined as

$$\chi_{\rm eff} \equiv \frac{m_1\chi_1 + m_2\chi_2}{M}. \tag{1.2}$$

Similarly, we use the effective precession parameter $\chi_p$ defined in [139],

$$\chi_p \equiv \frac{\max(A_1 S_{1,\perp}, A_2 S_{2,\perp})}{A_1 m_1^2}, \tag{1.3}$$

where $A_1 = 2 + 3/(2q)$ and $A_2 = 2 + 3/(2q)$ and $S_{i,\perp}$ is the magnitude of the in plane spins. That is, $S_{i,\perp} = |\boldsymbol{S_i} - \boldsymbol{S_i}\cdot\boldsymbol{L}|$, where $\boldsymbol{L}$ is the orbital angular momentum. We also introduce the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$.

We denote $\boldsymbol{\vartheta}$ as the set of all GW parameters including intrinsic parameters of the source and the extrinsic parameters related to projection of the waveform onto the detector. In the case of eccentric aligned-spin binaries we have 13 parameters $\boldsymbol{\vartheta} = \{\mathcal{M}, q, \chi_1, \chi_2, e, \zeta, \alpha, \delta, d_{\rm L}, t_{\rm coal}, \iota, \psi, \phi_{\rm coal}\}$. In the case of quasi-circular precessing-spin binaries we have 15 parameters $\boldsymbol{\vartheta} = \{\mathcal{M}, q, |\chi_1|, |\chi_2|, \theta_1, \theta_2, \phi_{\rm JL}, \phi_{12}, \alpha, \delta, d_{\rm L}, t_{\rm coal}, \theta_{\rm JN}, \psi, \phi_{\rm coal}\}$.

We indicate with $e$ the eccentricity and with $\zeta$ the relativistic anomaly, which are defined more precisely in Sec. III. The parameters $\{|\chi_1|, |\chi_2|, \theta_1, \theta_2, \phi_{\rm JL}, \phi_{12}\}$ describe the (dimensionless) spins values, spin tilts, the angle between the total and orbital angular momentum, and the angle between the spins of the BHs, respectively [140]. The rest are extrinsic parameters with $\alpha$ the right ascension, $\delta$ the declination, $d_{\rm L}$ the luminosity distance, $t_{\rm coal}$ the time of coalescence, $\theta_{\rm JN}$ the angle between the total angular momentum and the line of sight of the observer, $\iota$ the angle between the orbital angular momentum and the line of sight of the observer, $\psi$ the polarization angle between the GW and detectors, and $\phi_{\rm coal}$ the orbital reference phase at coalescence (e.g., see Figs. 1-3 in Ref. [141] for a diagram, and Eq. (17-20) in Ref. [142] for how the extrinsic parameters enter the detector projections).

In the following we assume the Planck 2015 cosmology [143].

## II. BAYESIAN INFERENCE

In this section, we outline how Bayesian inference can be used to determine if a detected GW event is more likely to be described by a BBH moving on an eccentric or quasi-circular orbit. The approach is to first compute posterior distributions for individual events, then to use these posteriors to compute Bayes' factors and finally weight these Bayes' factors by astrophysical information to calculate the probability that a BBH is eccentric.

### A. Bayes theorem, Bayes factor and odds ratios

Here, we summarize the basics of the Bayesian inference formalism used in GW astronomy. In doing so, we also introduce the concepts of Bayes factors and odds ratios.

Our hypothesis, $\mathcal{H}_a$, is that in the detector data, $d$, an observed GW signal is described by a waveform model, $a$, with parameters $\boldsymbol{\vartheta}_a$. The posterior probability distribution on the parameters of the model, $\boldsymbol{\vartheta}_a$, given the hypothesis, $\mathcal{H}_a$, is obtained using Bayes' theorem,

$$p(\boldsymbol{\vartheta}_a|d,\mathcal{H}_a) = \frac{p(d|\boldsymbol{\vartheta}_a,\mathcal{H}_a)\,p(\boldsymbol{\vartheta}_a|\mathcal{H}_a)}{p(d|\mathcal{H}_a)}\,. \tag{2.1}$$

Here, $p(\boldsymbol{\vartheta}_a|\mathcal{H}_a)$ is the prior probability distribution, $p(d|\boldsymbol{\vartheta}_a,\mathcal{H}_a)$ is the likelihood function, and $p(d|\mathcal{H}_a)$ is the evidence of the hypothesis $\mathcal{H}_a$. In this study, we compute most

of the posterior distributions with `DINGO` (see Sec. II C). We also compute a few posteriors with `pBilby` [134] for validation and comparison.

For a detector with stationary, Gaussian noise, we use the likelihood function

$$p(d|\boldsymbol{\vartheta}_a, \mathcal{H}_a) \propto \exp\left[-\tfrac{1}{2}\left\langle d - h_a(\boldsymbol{\vartheta}_a) | d - h_a(\boldsymbol{\vartheta}_a)\right\rangle\right], \tag{2.2}$$

where $h_a$ is the GW signal generated by waveform model $a$, and the brackets denote the noise-weighted inner product,

$$\langle A|B\rangle = 2\int_{f_{\text{low}}}^{f_{\text{high}}} df\, \frac{\tilde{A}^*(f)\tilde{B}(f) + \tilde{A}(f)\tilde{B}^*(f)}{S_n(f)}. \tag{2.3}$$

Here, $\tilde{A}(f)$ is the Fourier transform of $A(t)$, the asterisk denotes the complex conjugation and $S_n(f)$ is the one-sided power spectral density (PSD) of the detector. The integration limits $f_{\text{low}}$ and $f_{\text{high}}$ are set by the bandwidth of the detector's sensitivity.

When trying to estimate whether the GW event is eccentric or quasi-circular, we compute the odds ratio $\mathcal{O}_{a/b}$ between the two hypotheses, $\mathcal{H}_a$ and $\mathcal{H}_b$, which is given by

$$\mathcal{O}_{a/b} = \frac{p(\mathcal{H}_a)}{p(\mathcal{H}_b)}\frac{p(d|\mathcal{H}_a)}{p(d|\mathcal{H}_b)} \equiv \frac{R_a}{R_b}\,\mathcal{B}_{a/b}. \tag{2.4}$$

The first ratio in the product is the *priors odds*, which represent our belief about the universe prior to observing any GW data. It can be determined through astrophysical simulations or other non-GW observations.[1] The second term is the *Bayes factor*, which is the odds of the hypotheses given the data[2].

In this study, we consider the eccentric aligned-spin (EAS) vs quasi-circular aligned-spin (QCAS) odds-ratio ($\mathcal{O}_{\text{EAS/QCAS}}$), for which we need to compute the merger rate of eccentric events $R_{\text{E}}$ and quasi-circular events $R_{\text{QC}}$. We do not consider the eccentric aligned-spin vs quasi-circular precessing odds-ratio as to our knowledge no rate estimates (and thus prior odds) for this fraction exist in the literature.

### B. Probability a GW event is eccentric

To determine the odds ratio (2.4), one needs to include the priors odds, i.e., some astrophysical information. But the odds ratio does not account for the fact that we have analyzed many GW events, and therefore have information about the rates directly from GW observations. Therefore, we propose an alternate way to measure the probability that an event is eccentric. This method is analogous to the way that the LVK Collaboration reports the probability that a GW detection is of astrophysical origin, with the so-called $p_{\text{astro}}$ [131]. It relies on the formalism derived in Ref. [131]. This method has three advantages over just computing the odds ratio.

1. When computing the merger rates in Eq. (2.4), we are essentially drawing from a prior distribution, $p(R_{\text{E}}, R_{\text{QC}})$, which is motivated by astrophysics. However, the astrophysical prior has a large variance. With an increasing number of GW events, we can begin constraining the prior on the rates from observations.
2. We can naturally incorporate selection effects which are not included when considering the ratio between eccentric and quasi-circular BBH rates in Eq. (2.4).
3. We can compute the probability that eccentric events exist in the population of GWs analyzed so far without referring to a specific event.

We now review the formalism of Ref. [131] and show how it can be used to derive the probability that an event is eccentric. Let $\{d_i\}$ be the collection of strain data segments of GW triggers. We assign each event a flag, $g_i$ which is 1 if the event is eccentric and 0 if the event is quasi circular. Our goal is to obtain the probability distribution over the set of flags, $\{g_i\}$. Let $N$ be the number of events. According to Bayes' theorem, the posterior distribution over the flags is

$$\begin{aligned} &p(\{g_i\}, R_{\text{E}}, R_{\text{QC}} \,|\, \{d_i\}, N) \\ &= \frac{p(\{d_i\}\,|\,\{g_i\}, N, R_{\text{E}}, R_{\text{QC}})\, p(\{g_i\}, N, R_{\text{E}}, R_{\text{QC}})}{p(\{d_i\}, N)}. \end{aligned} \tag{2.5}$$

Let us recover an expression for the likelihood term, $p(\{d_i\}|\{g_i\}, N, R_{\text{E}}, R_{\text{QC}})$. The probability that the data segment $d_i$ is from an eccentric BBH is

$$Z_{\text{E}}(d_i) \equiv p(d|\mathcal{H}_{\text{E}}) = \int d\boldsymbol{\vartheta}\, p(d|\boldsymbol{\vartheta}, \mathcal{H}_{\text{E}}) p(\boldsymbol{\vartheta}|\mathcal{H}_{\text{E}}). \tag{2.6}$$

Here the "E" subscript indicates that an eccentric model is being used. Similarly, $Z_{\text{QC}}(d_i)$ is the probability that $d_i$ is generated by from a quasi-circular BBH. This is given by Eq. (2.6), but using the quasi-circular waveform, which we denote as "QC". Thus, we can write the likelihood as

$$p(\{d_i\}\,|\,\{g_i\}, N, R_{\text{E}}, R_{\text{QC}}) = \left[\prod_{\{i|g_i=1\}}^{N_{\text{E}}} Z_{\text{E}}(d_i)\right]\left[\prod_{\{i|g_i=0\}}^{N_{\text{QC}}} Z_{\text{QC}}(d_i)\right]. \tag{2.7}$$

Here, the conditioning on the rates does not enter the expression for the rate likelihood. Note that we are only using aligned-spin evidences in the above rate likelihood and not considering spin-precessing evidences. We leave extending these results to the spin-precessing case for future work.

The prior distribution in Eq. (2.5) can be factorized exactly the same way as Eqs. (13)–(17) in Ref. [131]. For completeness, we include the final expression here:

$$p(\{g_i\}, R_{\text{E}}, R_{\text{QC}}, N) = R_{\text{E}}^{N_{\text{E}}} R_{\text{QC}}^{N_{\text{QC}}} \frac{e^{-(R_{\text{E}}+R_{\text{QC}})}}{N!} p(R_{\text{E}}, R_{\text{QC}}), \tag{2.8}$$

[1] If only analyzing a subset of GW events, we could also use the rate estimates from previous GW observing runs to determine the rates (and thus) prior odds.

[2] Note these Bayes factors are defined differently from Ref. [20] and cannot directly be compared. The Bayes factor in Eq. (2.4) will be conservative (smaller) compared to the $e_{10\text{Hz}} < 0.05$ vs $e_{10\text{Hz}} > 0.05$ Bayes factor considered in Ref. [20].

where $N_{\rm E} = \sum_i g_i$ and $N_{\rm QC} = \sum_i (1-g_i) = N - N_{\rm E}$. The last term in Eq. (2.8) is the prior over the rates. This can either be a uniform prior or an astrophysical prior from the literature. We discuss the derivation of an astrophysical prior on the rates in Sec. VI A. The exponential term comes from the Poisson uncertainty on the number of events [122]. Substituting the likelihood and prior term into Eq. (2.5) our posterior over the flags and rates becomes:

$$\begin{aligned} &p(\{g_i\}, R_{\rm E}, R_{\rm QC} \,|\, \{d_i\}, N) \\ &\quad \propto \left[\prod_{\{i|g_i=1\}}^{N_{\rm E}} R_{\rm E} Z_{\rm E}(d_i)\right]\left[\prod_{\{i|g_i=0\}}^{N_{\rm QC}} R_{\rm QC} Z_{\rm QC}(d_i)\right] \\ &\quad \times e^{-(R_{\rm E}+R_{\rm QC})} p(R_{\rm E}, R_{\rm QC}), \end{aligned} \tag{2.9}$$

where we have dropped the evidence term and the factorial, and replaced them with a proportionality. If we are only interested in an estimate for the rates, we can marginalize over the flags to get

$$\begin{aligned} &p(R_{\rm E}, R_{\rm QC} \,|\, \{d_i\}, N) \\ &\propto \left[\prod_i R_{\rm E} Z_{\rm E}(d_i) + R_{\rm QC} Z_{\rm QC}(d_i)\right] e^{-(R_{\rm E}+R_{\rm QC})} p(R_{\rm E}, R_{\rm QC}). \end{aligned} \tag{2.10}$$

To obtain the normalization of Eq. (2.10) we can numerically integrate over $R_{\rm E}$ and $R_{\rm QC}$.

Finally, we can compute the probability that the $m$'th event is eccentric by marginalizing over the rates and all flags except the $m$'th flag. This results in the expression

$$\begin{aligned} &p_{\rm ecc}(m) = p(g_m = 1 \,|\, \{d_i\}, N) \\ &= \int dR_{\rm E}\, dR_{\rm QC}\, R_{\rm E} Z_{\rm E}(d_m) \frac{p(R_{\rm E}, R_{\rm QC}|\{d_i\}, N)}{R_{\rm E} Z_{\rm E}(d_m) + R_{\rm QC} Z_{\rm QC}(d_m)}. \end{aligned} \tag{2.11}$$

We can gain intuition on this statistic by considering a limiting case which relates to the odds ratio. Suppose that the set of GW observations holds no information on $R_{\rm E}$ and $R_{\rm QC}$. This means that $p(R_{\rm E}, R_{\rm QC}|\{d_i\}, N) = p(R_{\rm E}, R_{\rm QC})$. We can define $(\tilde{R}_{\rm E}, \tilde{R}_{\rm QC}) = \text{argmax}\{p(R_{\rm E}, R_{\rm QC})\}$. That is, the pair $(\tilde{R}_{\rm E}, \tilde{R}_{\rm QC})$ are the merger rates for which the function $p(R_{\rm E}, R_{\rm QC})$ is maximized. In this context, $\tilde{R}_{\rm E}$ and $\tilde{R}_{\rm QC}$ are the rates we would use to estimate the odds ratio in Eq. (2.4). If we draw one sample from $p(R_{\rm E}, R_{\rm QC})$ to estimate Eq. (2.11) we will find the most likely estimate for $p_{\rm ecc}$ is:

$$\begin{aligned} \tilde{p}_{\rm ecc}(m) &\equiv \frac{\tilde{R}_{\rm E} Z_{\rm E}(d_m)}{\tilde{R}_{\rm E} Z_{\rm E}(d_m) + \tilde{R}_{\rm QC} Z_{\rm QC}(d_m)} \\ &= \frac{\mathcal{O}_{\rm EAS/QCAS}}{1 + \mathcal{O}_{\rm EAS/QCAS}}. \end{aligned} \tag{2.12}$$

Thus, if we use the prior instead of the rate posterior in Eq. (2.11), $p_{\rm ecc}$ can be thought of as a mapping of the odds ratio to the interval [0, 1]. As more events are observed, $p(R_{\rm E}, R_{\rm QC}|\{d_i\}, N)$ becomes tighter than $p(R_{\rm E}, R_{\rm QC})$. This means we lose the large uncertainty due to the astrophysical uncertainty on the rates and get a better estimate on $p_{\rm ecc}$.

We can also compute the probability that there exists at least one eccentric event in the population of GW events analyzed so far. This can be done by defining

$$p_{\rm ecc,\,pop} \equiv 1 - p(\{g_i = 0\} \,|\, \{d_i\}, N). \tag{2.13}$$

That is, the probability there exists eccentricity in the population is one minus the probability that all events are quasi-circular. This can be computed by marginalizing Eq. (2.9) over the rates and normalizing by the integral over the rates and flags. Explicitly:

$$\begin{aligned} &p(\{g_i = 0\} \,|\, \{d_i\}, N) = \\ &\frac{\int dR_{\rm E}\, dR_{\rm QC}\, R_{\rm QC}^N \prod_i Z_{\rm QC}(d_i) e^{-(R_{\rm E}+R_{\rm QC})} p(R_{\rm E}, R_{\rm QC})}{\int dR_{\rm E}\, dR_{\rm QC} \prod_j \left[R_{\rm QC} Z_{\rm QC}(d_j) + R_{\rm E} Z_{\rm E}(d_j)\right] e^{-(R_{\rm E}+R_{\rm QC})} p(R_{\rm E}, R_{\rm QC})}. \end{aligned} \tag{2.14}$$

To compute the normalizing factor (denominator) exactly we need to integrate over a product with many terms. We can instead compute the dominant terms of this integral to place a lower bound on the normalizing factor. This allows us to place an upper bound on $p(\{g_i = 0\} \,|\, \{d_i\}, N)$ thereby allowing us to place a lower bound on $p_{\rm ecc,\,pop}$. We discuss a method to compute the dominant terms in this integral in Appendix B.

Finally, we can incorporate selection effects into this framework. Selection effects occur because we do not observe every BBH in our prior. Low mass or high distance events may not have a high enough signal-to-noise ratio (SNR) to be seen by ground-based detectors. This has two effects on the likelihood in Eq. (2.7), which are derived in Ref. [144].

We define $\alpha_{\rm E}$ and $\alpha_{\rm QC}$ as the fraction of events in the universe which would be detected under the eccentric and quasi-circular hypotheses respectively. Explicitly, $\alpha_E$ can be computed as:

$$\alpha_{\rm E} = \int d\boldsymbol{\vartheta}\, p_{\rm E,\,det}(\boldsymbol{\vartheta}) p(\boldsymbol{\vartheta}|\mathcal{H}_E). \tag{2.15}$$

where $p_{\rm E,\,det}(\boldsymbol{\vartheta})$ is the probability of detecting a GW event with parameters $\boldsymbol{\vartheta}$ under the eccentric hypothesis. The computation for $\alpha_{\rm QC}$ is analogous except using $p_{\rm QC,det}(\boldsymbol{\vartheta})$ (the probability to detect an event with parameters $\boldsymbol{\vartheta}$ under the quasi-circular hypothesis). We can compute $p_{\rm QC,det}$ and $p_{\rm E,det}$ using a threshold SNR and error functions [131].

In general, $p_{\rm E,det}$ and $p_{\rm QC,det}$ are different. This is because eccentric waveforms are not usually used for GW searches (see however, Refs. [145–148]). To compute $p_{\rm E,det}$, to a high accuracy one would need to perform a suite of synthetic eccentric injections into the LVK detection pipelines and count the number of observed triggers [13]. Alternatively, one could compute the overlap between the eccentric signal and a quasi-circular waveform maximizing over the parameters of the system. This was done in Ref. [31] for an eccentric non-spinning fiducial injection with $d_L$ =1Gpc and $\mathcal{M} = 28 M_\odot$ using the waveform model from Refs. [149–151]. The authors of Ref. [31] find that the deviation between $p_{\rm E,det}$ and $p_{\rm QC,det}$ is less than 0.05 at $e_{\rm 10Hz}$ for this injection. However, we must exercise caution as $p_{\rm E,det}$ and $p_{\rm QC,det}$ will be different for different injection configurations. One could imagine GWs whose

SNR is near the boundary of the threshold SNR, for detection causing larger differences between $p_{\rm E,det}$ and $p_{\rm QC,det}$. However, as a detailed computation of $p_{\rm E,det}$ has not been done, we use the approximation $p_{\rm E,det} = p_{\rm QC,det}$ for this paper and leave a more robust computation for future work.

According to Ref. [144] to incorporate selection effects we replace $R_{\rm E}$ with $\alpha_{\rm E} R_{\rm E}$ and $R_{\rm QC}$ with $\alpha_{\rm QC} R_{\rm QC}$. We need to also replace $Z_{\rm E}$ with $Z_{\rm E}/\alpha_{\rm E}$ and $Z_{\rm QC}$ with $Z_{\rm QC}/\alpha_{\rm QC}$. This amounts to replacing $\exp(R_{\rm E} + R_{\rm QC})$ with $\exp(\alpha_{\rm E} R_{\rm E} + \alpha_{\rm QC} R_{\rm QC})$ in Eqs. (2.7)–(2.10). Thus our posterior over the rates including selection effects is:

$$\begin{aligned} p(\{g_i\}, &R_{\rm E}, R_{\rm QC}, |\{d_i\}, N) \\ &\propto \left[\prod_{\{i|g_i=1\}}^{N_{\rm E}} R_{\rm E} Z_{\rm E}(d_i)\right]\left[\prod_{\{i|g_i=0\}}^{N_{\rm QC}} R_{\rm QC} Z_{\rm QC}(d_i)\right] \\ &\times e^{-(\alpha_{\rm E} R_{\rm E} + \alpha_{\rm QC} R_{\rm QC})} p(R_{\rm E}, R_{\rm QC}). \end{aligned} \tag{2.16}$$

This can then be propagated into Eqs. (2.10) and (2.11).

### C. DINGO

We now summarize the main features of the machine-learning code DINGO [152–154]. In particular, its use of normalizing flows[155, 156] and importance sampling [115, 157].

In many practical applications, obtaining $p(\boldsymbol{\vartheta}|d)$ in Eq. (2.1) is analytically impossible. But one can use techniques such as nested sampling [158] or Markov-Chain Monte-Carlo (MCMC) [159] to sample from $p(\boldsymbol{\vartheta}|d)$. The LVK Collaboration has developed tools such as `LALInference` [160] and `Bilby` [161–163] for this task. However, these methods must evaluate the likelihood $O(10^{7-8})$ times per event. Thus, for time-domain waveform models produced by solving ordinary differential equations parameter estimation can be expensive. For example, `SEOBNRv4EHM` takes $O(100-700\text{ms})$ per likelihood evaluation meaning inference can take $O(\text{week})$ per event even parallelizing over 320 cores [24]. It is then very computationally expensive to analyze the catalog of GW events with such waveform model.

We can instead use likelihood-free approaches to amortize the inference [153, 164, 165]. Here we use `DINGO` [153], which has been shown to achieve results with the same accuracy as standard samplers [115]. We pay the upfront cost of training a neural network for $O(\text{week})$, but then we can do inference on any events within the trained priors in $O(\text{hour})$.

`DINGO` learns a mapping $f_{d,S_n} : u \to \boldsymbol{\vartheta}$ from a simple-base distribution $p(u) = \mathcal{N}(0,1)^D$ to the complex target GW posterior $p(\boldsymbol{\vartheta}|d)$ (it is implied this is also conditioned on $\mathcal{H}_a$). This mapping is learned through a series of composable functions parameterized by neural networks (a normalizing flow). For each data sample $d$, the forward model is given by

$$\begin{aligned} q(\boldsymbol{\vartheta}|d, S_n) &= [f_{d,S_n}]_* \mathcal{N}(0,1)^D \\ &= \mathcal{N}(0,1)^D\left(f^{-1}_{d,S_n}(\boldsymbol{\vartheta})\right)\left|\frac{\partial f^{-1}_{d,S_n}}{\partial \boldsymbol{\vartheta}}\right|, \end{aligned} \tag{2.17}$$

where $D$ is the number of GW parameters and $(\cdot)_*$ denotes the push-forward operator. The $d$ and $S_n$ on the subscript of $f_{d,S_n}$ indicate that the mapping uses $d$ and $S_n$ as context to the neural network. The Jacobian in the last term of Eq. (2.17) is a normalization factor to account for the fact that in general $f_{d,S_n}$ is not a volume preserving operation [156].

`DINGO` is unique from other flow approaches as the inference is aided by conditioning the networks on the arrival times of GWs in the interferometers. We train two separate networks: one to learn the distribution $q(t_i|d)$ and one to learn the distribution $q(\boldsymbol{\vartheta}|d, S_n, \hat{t}_i)$. Here, $t_i$ are the arrival times of the GW in each detector and can be computed from $\boldsymbol{\vartheta}$. Then $\hat{t}_i$ are "blurred" versions of the arrival times. That is, $p(\hat{t}_i|t_i) = \text{Unif}(t_i - 1\text{ms}, t_i + 1\text{ms})$. The process is to start with an initial guess for $t_i \sim q(t_i|d)$. Then using $t_i$, sample $\hat{t}_i \sim p(\hat{t}_i|t_i)$ and get an initial guess for $\boldsymbol{\vartheta} \sim q(\boldsymbol{\vartheta}|d, S_n, \hat{t}_i)$. After obtaining initial guesses, we can Gibbs sample [166] the distributions $p(\hat{t}_i|t_i)$ and $q(\boldsymbol{\vartheta}|d, S_n, \hat{t}_i)$ to obtain $q(\boldsymbol{\vartheta}, \hat{t}_i|d, S_n)$. Finally, we can drop $\hat{t}_i$ from $q(\boldsymbol{\vartheta}, \hat{t}_i|d, S_n,)$ to obtain $q(\boldsymbol{\vartheta}, |d, S_n)$. The advantage in this process is that the network learning $q(\boldsymbol{\vartheta}|d, S_n, \hat{t}_i)$ never sees a a GW with the merger greater than 1ms away from the start of the data segment. To summarize, by conditioning on the blurred arrival times, we standardize the data, allowing for smoother training.

`DINGO` also uses an embedding network to compress the high dimensional Fourier domain strain data into a feature vector of length 128 for each detector. This embedding network is trained as part of the mapping $f_{d,S_n}$.

After performing inference with the network, we need to have a guarantee that the distribution $q(\boldsymbol{\vartheta}|d, S_n)$ produced by `DINGO`, is a good approximation to the true posterior distribution, $p(\boldsymbol{\vartheta}|d, S_n)$. To achieve this, we can importance sample the generated proposal distribution to obtain samples from the true distribution [157]. Explicitly, we first sample $\boldsymbol{\vartheta}_i \sim q(\boldsymbol{\vartheta}|d, S_n)$ and then weight each sample according to

$$w(\boldsymbol{\vartheta}_i) \propto \frac{p(d|\boldsymbol{\vartheta}_i, S_n) p(\boldsymbol{\vartheta}_i)}{q(\boldsymbol{\vartheta}|d, S_n)}. \tag{2.18}$$

Here we only need to evaluate the likelihood for as many points as there are in the proposal posterior, which is orders of magnitude lower than a Nested Sampling or MCMC run. Since we only need to compute the likelihood during importance sampling, this can be parallelized across an arbitrary number of cores. In addition, since `DINGO` is trained on a forward Kullback-Leibler (KL) divergence objective, it tends to cover regions of high density. This is preferred in scientific domains, as we want to ensure a conservative as possible estimate before importance sampling.

The *effective sample size*, $n_{\rm eff}$, can be computed from the importance weights using:

$$n_{\rm eff} = \frac{(\sum_i w_i)^2}{\sum_i w_i^2}. \tag{2.19}$$

This represents the number of samples drawn from the true posterior distribution $p(\boldsymbol{\vartheta}|d)$ [157]. Thus, any test statistic computed with the weighted samples is equivalent to using $n_{\rm eff}$

samples from the true posterior. The importance weights can also be used to compute the evidence, $p(d)$, (see Eqs. (2)—(8) in Ref. [115]). When reporting results with importance sampling, we label the result `DINGO-IS`.

Finally, in this paper we often compute the Jensen-Shannon divergence (JSD) between posterior distributions as a comparison statistic. For example, this is used to compare `DINGO-IS` with `pBilby`. The JSD is defined as a symmetrized version of the KL divergence that is between 0 and 1 bits. It is defined as a divergence between two distributions $p(\boldsymbol{\vartheta})$ and $q(\boldsymbol{\vartheta})$:

$$D_{\rm JS} = \frac{D_{\rm KL}(p|q) + D_{\rm KL}(q|p)}{2}, \tag{2.20}$$

where the KL divergence is

$$D_{\rm KL}(p|q) = \int d\boldsymbol{\vartheta}\, p(\boldsymbol{\vartheta}) \log_2\left(\frac{p(\boldsymbol{\vartheta})}{q(\boldsymbol{\vartheta})}\right). \tag{2.21}$$

The KL divergence expresses the information loss we would accumulate if using distribution $p$ to approximate a test statistic when $q$ was the true underlying distribution.

## III. ECCENTRIC AND QUASI-CIRCULAR WAVEFORM MODELS

In this paper we use mostly multipolar eccentric and quasi-circular aligned-spin waveform models, but for some applications also quasi-circular spin-precessing models. Several non-spinning and aligned-spin, eccentric inspiral-merger-ringdown models have been proposed in the literature [149, 167–175]. Here, we employ the (time-domain) eccentric aligned-spin model `SEOBNRv4EHM`, developed in Refs. [119, 120], the quasi-circular aligned-spin model `SEOBNRv4HM` [117, 118], and the quasi-circular spin-precessing model `SEOBNRv4PHM` [118, 125–128]. For some studies, we also use the quasi-circular spin-precessing NR surrogate model `NRSur7dq4` [129].

The (EOB) formalism [125, 135, 176–178] maps the two-body dynamics onto the one of a test body in a deformed Kerr metric, the deformation being the symmetric mass ratio $\nu$. It relies on three key ingredients: (i) a Hamiltonian that describes the conservative dynamics, (ii) a radiation-reaction force, and (iii) the gravitational modes. These three components are built by resumming PN calculations. Furthermore, the EOB approach also provides the full inspiral-merger-ringdown waveforms using physically motivated ansatze for the merger, and results from BH perturbation theory for the ringdown. Finally, EOB waveforms are made highly accurate through calibration to NR simulations [128, 179–182], and more recently from the gravitational self-force approach [183].

To fully characterize GWs from eccentric BBHs, one needs to include two additional parameters compared to the quasi-circular case. The `SEOBNRv4EHM` [24, 119, 120] model adds two new parameters to the quasi-circular spin-aligned model, `SEOBNRv4HM` [117, 118]: the initial orbital eccentricity $e$ and relativistic anomaly $\zeta$. The model employs the quasi-Keplerian parametrization to express the relationship between the radial separation $r$ and these parameters [24, 119]

$$r = \frac{p}{1 + e\cos\zeta}, \tag{3.1}$$

where $p$ is the semi-latus rectum. In order to integrate the EOB Hamilton equations, one needs to specify initial conditions (see Eqs. (12)–(14) in Ref. [24]). In `SEOBNRv4EHM` the eccentricity and relativistic anomaly are part of these initial conditions, and they are specified at an orbit-averaged frequency. We note that one could specify the initial conditions at an instantaneous reference frequency. However, this strongly affects the time to merger and creates a rapidly varying log-likelihood surface, which can cause difficulties for parameter-estimation pipelines, as pointed out in Ref. [24]. In addition, unlike the instantaneous frequency, the orbit-averaged GW frequency is related to twice the orbit-averaged orbital frequency [133] for eccentric orbits. Thus we can use it to uniquely specify the point along the orbit where we measure the eccentricity and relativistic anomaly. We denote the quasi-Keplerian eccentricity and relativistic anomaly measured at orbit-average frequency of 10Hz in the detector frame as $e_{\rm 10Hz}$ and $\zeta_{\rm 10Hz}$.

The dynamics of the BBH is used to construct the waveform modes, $h_{\ell m}$. The `SEOBNRv4EHM` model contains the $(\ell,|m|) = (2,2),(2,1),(3,3),(4,4),(5,5)$ multipoles. The inspiral waveform modes contain 2PN eccentric corrections derived in Ref. [120], and are enhanced during the late inspiral and plunge with non-quasi-circular terms that are fitted to quasi-circular NR simulations (see Eqs. (7)–(10) in Ref. [119]). Note that in this model the radiation-reaction force does *not* have eccentric corrections, and the merger-ringdown modes are assumed to be quasi-circular [118]. This latter assumption is used by all the current eccentric waveform models, and it is supposed to hold for mild eccentricities [112, 184, 185].

The waveform modes can be used to construct the gravitational polarizations:

$$h_+ - ih_\times = \sum_{l=2}^{\infty}\sum_{m=-l}^{m=l} {}_{-2}Y_{lm}(\varphi,\iota)h_{lm}(m_{1,2},\chi_{1,2},e,\zeta;t), \tag{3.2}$$

where for aligned-spins $(\varphi,\iota)$ are the azimuthal and polar angles to the observer in the source frame. Without loss of generality, $\varphi = \phi_{\rm coal}$ [186].

We remark that the `SEOBNRv4EHM` model has been tested against (mainly) non-spinning NR simulations up to $e_{\rm 20Hz} = 0.3$ [133], finding good agreement. Since we are generating waveforms with a starting frequency of 10Hz, we use [41] to map the maximum allowed eccentricity to $e_{\rm 10Hz} \leq 0.5$. Thus, we cannot say anything about the accuracy of the waveform model for events that have eccentricities greater than $e_{\rm 10Hz} > 0.5$. We stress also that `SEOBNRv4EHM` does not simultaneously model spin-precession and eccentricity[3] and thus

[3] This is also true of other inspiral-merger-ringdown waveform models in the literature with the exception of Ref. [130].

we cannot address the possibility that an event is both eccentric and precessing.

Regarding the quasi-circular spin-precessing waveform model `SEOBNRv4PHM` [118, 128], it integrates the time-domain EOB dynamics in a co-precessing frame using the EOB spin-precession equations and a Hamiltonian calibrated to aligned-spin NR simulations. In such a frame, it contains the $(\ell,|m|) = (2,2),(2,1),(3,3),(4,4),(5,5)$ multipoles. Then, the waveform in the co-precessing frame is rotated to the inertial frame [126, 127]. In contrast, `NRSur7dq4` directly interpolates 1528 precessing NR waveforms with mass ratios $q \geq 1/4$ with $\chi_{1,2} \leq 0.8$ [187]. The model can also be used in the extrapolation region with $q \geq 1/6$ and $\chi_{1,2} \leq 1$. This model is typically used for heavier mass systems in parameter estimation studies as it is restricted by the length of the training waveforms. This model includes all $\ell \leq 4$ modes.

Before we end the section, we briefly comment on the eccentricity parameter. Since eccentricity is not uniquely defined in GR there are multiple ways to parametrize the orbit [188]. Waveform models may use orbital parameters, compact object trajectories, energy and angular momentum or other parameterizations to define eccentricity [119, 120, 149, 175, 189–199].

It is, however, possible to use a definition of eccentricity that can be extracted from the waveform. This was proposed in Refs. [132, 133], where this new eccentricity is denoted $e_{\rm gw}$. This $e_{\rm gw}$ is extracted by interpolating the instantaneous GW frequency of the (2,2) mode, $\omega_{22}$, along the pericenter and apocenter points. This is calculated as

$$e_{\rm gw} = \cos(\psi/3) - \sqrt{3}\sin(\psi/3), \tag{3.3}$$

with

$$\psi = \arctan\left(\frac{1-e_{\omega_{22}}^2}{2e_{\omega_{22}}}\right), \tag{3.4}$$

and

$$e_{\omega_{22}} = \frac{\omega_{22,p}^{1/2} - \omega_{22,a}^{1/2}}{\omega_{22,p}^{1/2} + \omega_{22,a}^{1/2}}, \tag{3.5}$$

where $\omega_{22,p}$ and $\omega_{22,a}$ are the GW frequency of the 22 mode at periastron and apocenter.

We can similarly interpolate the GW mean anomaly

$$l_{\rm gw} = 2\pi\frac{t-t_i^p}{t_{i+1}^p - t_i^p}. \tag{3.6}$$

Here $t$ is the time at which we are measuring the mean anomaly and $t_i$ is the time of the $i$th periastron passage measured using the (2,2) mode frequency.

We compute $e_{\rm gw}$ using parallelized version of the `gw_eccentricity` package [24, 132] [4] . We choose to measure the eccentricity at a dimensionless frequency of $M_{\rm det} f_{\rm ref}$, where $M_{\rm det}$ is the detector frame total mass of the system and $f_{\rm ref} = 10$ Hz. The reason to multiply by the total mass is because we do not want our reported eccentricity to depend on the redshift of the source. When reporting the GW eccentricity at 10Hz we denote this as $e_{\rm gw,10Hz}$.

[4] We use the "Amplitude" method in `gw_eccentricity` and set `num_orbits_to_exclude_before_merger = 1`

## IV. ZERO-NOISE SYNTHETIC-DATA AND REAL EVENTS

### A. Trained networks and priors

We train twenty-five `DINGO` networks on 80GB NVIDIA-A100 GPUs each for 11 days. We train eleven networks for the eccentric aligned-spin case, eleven networks for quasi-circular aligned-spin case and four networks for the quasi-circular precessing-spin case. The number of networks corresponds to the fact that one has to train a separate network for each observing run (noise curve), detector configuration, waveform model, choice of gravitational modes, mass prior and distance prior. The mass and distance prior range is optional, but we find that training two separate networks for the regimes $d_{\rm L} < 3$ Gpc and $d_{\rm L} < 6$ Gpc leads to increased sample efficiency. Similarly, if we train a network with a smaller mass prior we achieve higher sample efficiency in the prior. But we also want to analyze heavy BBHs so we train a separate network increasing the upper bound of the mass prior. Since there are some GW events that only occur in one detector in the third-observing run, we also train networks for LIGO-Hanford and LIGO-Livingston only. This is also useful for doing single-detector analyses of events with glitches. We train the networks in the range $(f_{\rm min}, f_{\rm max}) = (20{\rm Hz}, 1024{\rm Hz})$.

While we do train a large number of networks, it is important to realize that the inference is amortized. This means the computational speed-up compared to traditional samplers will scale as the number of events increases. For example, in O1 where we only analyze only one GW event, there is no speed-up compared to a traditional sampler. However, in O3 where we analyze 51 events, the speed-up is on the order of 6 months. Another subtle point is that in GW science, we rarely perform parameter estimation for each event once. We may want to change the prior, $f_{\rm min}$, $f_{\rm max}$ or in the case of a traditional sampler boost the sampler settings. We also may want to study maximum likelihood injections for consistency. For example, in this study, we obtained over 300 posteriors across different waveform models before generating the final catalogue of 57 events. Thus, while we train a large number of networks, the speed-up is still significant and scales well with a large number of GW events.

We now discuss the subset of events we do not analyze. We do not analyze events with a chirp mass below 15 $M_\odot$. This includes neutron-star events (although this is being addressed in an upcoming work [200]). This is because low-mass systems have rapid Fourier-domain oscillations at low frequencies. As a consequence, if we want to represent the data, we need to use a much larger embedding network, which leads to significant computational cost. As a low chirp mass corresponds to a longer signal, this constraint implies we do not analyze events

longer than 16 seconds.

We also do not report on events for which we were unable to obtain 5,000 effective samples. Note that we have purposely set this to a very high threshold in order to be indistinguishable from other samplers like `pBilby`, and have a smooth posterior. We see that reducing this threshold leads to spikes in the posterior density, which can lead to JSDs with `pBilby` larger than 0.002 bits (the expected stochastic deviation from identical nested sampling runs [162]).

We set an isotropic in component spin prior with $|\chi_{1,2}| < 0.9$. We also set a uniform prior in $\zeta_{10\mathrm{Hz}} \in [0, 2\pi]$, a uniform prior in $e_{10\mathrm{Hz}} \in [0, 0.5]$ and a uniform in component mass prior on the mass ratio $q \in [0.125, 1]$. The upper bound in the $e_{10\mathrm{Hz}}$ is set by the restriction of the waveform model (see Sec. III). Other than this, the priors are the same as in Ref. [115].

We use a uniform prior on eccentricity as opposed to a log-uniform prior informed by astrophysics. We instead incorporate the astrophysics information by folding it into our computation of the prior odds (see Eq. (2.4)) or $p_{\mathrm{ecc}}$ (see Eq. (2.11)). Having a uniform prior is also useful because if we perform a population study in the future, we can importance sample to a multitude of different priors [13]. In contrast, if we use a light-tailed log-uniform prior, we could not easily convert to a different prior. This is because there would be many high-weight points at high eccentricities leading to low sample efficiency. A separate reason to adopt a uniform prior instead of an astrophysical log-uniform one is so that we do not have to decide where to place the lower bound of a log-uniform prior. Changing this lower bound can strongly affect Bayes factors as it gives high weight to low eccentricities in the evidence integral.

However ultimately, a feature of a good Bayesian analysis is using multiple reasonable priors. For completeness, we also compute Bayes factors with a log-uniform prior on $e_{10\mathrm{Hz}} \in (10^{-4}, 0.5)$ for GW200129, GW190701 and GW200208_22. We display the results in Table I and in Fig. 2. We note that because these events are likelihood dominated, there is very little shift in the posterior, but the Bayes factors for eccentricity do decrease.

When performing the analysis with the quasi-circular spin-precessing `SEOBNRv4PHM` and `NRSur7dq4` models, we utilize the same priors as in the previous paragraph except for the dimensionless spin magnitudes. As stated above, for the eccentric model we restrict the dimensionless spin magnitudes to $|\chi_{1,2}| < 0.9$. However, in the case of GW200129 with `NRSur7dq4`, there is support for $a_{1,2} > 0.9$. Thus we perform the spin-precessing analysis with and without the spin-prior restriction and take the higher evidence. This is to avoid artificially lowering the evidence for precession [201]. For the spin angles we use the same priors as in Ref. [115].

### B. Zero-noise synthetic injection

To validate the use of `DINGO` with eccentric waveform models, we first perform two synthetic-signal (injection) in zero-noise using two independent samplers, `pBilby` and `DINGO`. Due to the expensive nature of doing `SEOBNRv4EHM` runs with `pBilby`, we use only the $(\ell, |m|) = (2, 2)$ modes in the injection and recovery.

We need to make some modifications to allow `DINGO` to work with zero noise. `DINGO` is trained on Gaussian detector noise. Thus we cannot simply inject a zero-noise signal into the data, as it will be flagged as out of distribution, and give a low effective sample size. Instead, we first run `DINGO` on 100 Gaussian noise realizations while fixing the injected parameters. We then pool the results. The idea is that when we average the `DINGO` posterior over infinite noise realizations, this should be close (though generally broader) than the true zero-noise posterior.

However, with this method we no longer have an estimate for the probability density of each sample as there is no guarantee the set of pooled samples are drawn from a normalized distribution. Thus, we train an unconditional density estimator on the set of pooled samples in order to recover the density of $q_{\mathrm{ZN}}(\boldsymbol{\vartheta}|h(f))$. We use one million pooled samples as a training dataset for the unconditional density estimator.

Now that we have the density of each sample, we importance sample the set of pooled samples, but using a zero noise likelihood, $p_{\mathrm{ZN}}(h|\boldsymbol{\vartheta})$, when computing the importance weights. Thus, we have an exact reconstruction of the zero-noise posterior distribution.

We present the analysis of two injections in Fig. 1 (note the result which should be used for comparison is the orange curve labeled `DINGO-IS`). The injection parameters are $\chi_{\mathrm{eff}} = -0.23$, $\mathcal{M}_{\mathrm{det}} = 28 M_\odot$, $q = 1/3$ and $e = 0.1$ or $e = 0.2$. The JSD between the 1D marginals of `pBilby` and `DINGO-IS` are all less than 0.002 bits (the expected stochastic variation from GW sampling algorithms [162]). We note that while `DINGO` and `pBilby` have a disagreement, after importance sampling this disagreement disappears. We can also see that the proposal density generated by `DINGO` is broader than the true posterior, a desirable property for importance sampling. These two facts give us confidence that `DINGO-IS` and `pBilby` have comparable results. We comment that for both samplers, there is some bias in the recovered chirp mass and eccentricity, which is reminiscent of the behavior in Ref. [98].

### C. GW200129

There has been considerable interest around GW200129 due to the fact that it has shown signs of orbital precession [201], evidence for a measurable kick velocity [203], false violations of GR due to waveform systematics [25], and a glitch [204, 205]. Therefore, to analyze this event we follow the treatment of Refs. [201, 204, 205]. Namely, we incorporate multiple glitch variations and analyze this event also for precession using both `SEOBNRv4PHM` and `NRSur7dq4`.

When analyzed with `SEOBNRv4EHM`, GW200129 is an intermediate-mass BBH consistent with zero effective spin. In particular, We find $M_{\mathrm{src}} = 78.3^{+4.2}_{-3.1}$, $M_{\mathrm{det}} = 69.4^{+4.2}_{-3.1}$ and $\chi_{\mathrm{eff}} = 0.02^{+0.12}_{-0.11}$. We measure $e_{\mathrm{gw,\,10Hz}} = 0.27^{+0.10}_{-0.12}$ and $\log_{10}\mathcal{B}_{\mathrm{EAS/QCAS}}$ in the range 1.84-4.75.

There is a striking variation in the Bayes factors when using different glitch mitigation techniques. In particular,

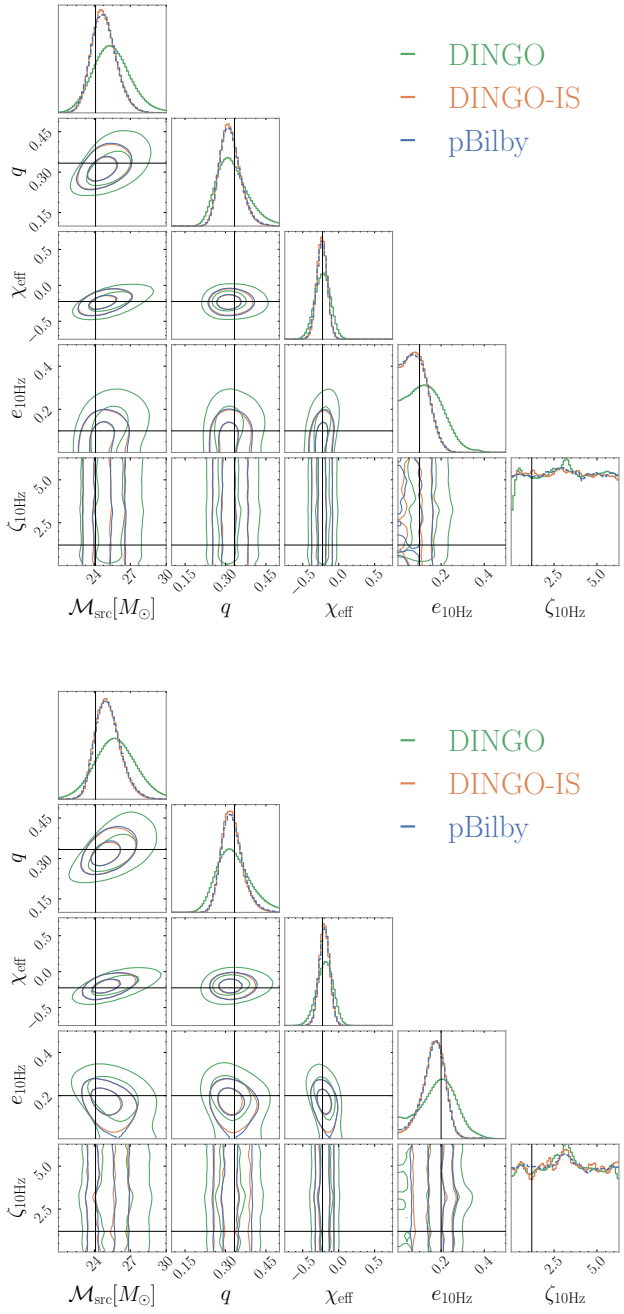


Figure 1. Zero-noise injections of `SEOBNRv4EHM` with parameters $\chi_{\rm eff} = -0.23$, $\mathcal{M}_{\rm det} = 28M_\odot$, $q = 1/3$ and initial eccentricities $e_{\rm 10Hz} = 0.1$ and $e_{\rm 10Hz} = 0.2$ for the top and bottom figures, respectively. The injections are recovered with both `DINGO`, `DINGO-IS` (`DINGO` with importance sampling) and `pBilby`.

`gwsubtract` [204, 206] favors eccentricity over `BayesWave` [207, 208] by 2+ orders of magnitude (see Table I and Figs. 2, 3 and 6). This makes interpreting the analysis challenging as one has to decide which glitch mitigation technique to trust.

The `gwsubtract` mitigation technique uses a witness time series from an auxillary channel in the interferometers to linearly subtract the noise. In particular one uses the witness time series and the strain data to compute a transfer function between the auxillary channel and true strain [206]. This transfer function is then used to estimate the de-glitched strain data. This method was used by the LVK Collaboration to mitigate the glitch in LIGO-Livingston during the official analysis. This is because the witness function was well estimated by the modulation control system at the time of the detection (see Sec. 3 in Ref. [204]). We also note that there have recently been additional studies of the impact of glitch mitigation on GW200129 in Refs. [209, 210] indicating that `gwsubtract` may under-subtract the glitch.

`BayesWave` on the other hand, takes a data driven approach. It models the astrophysical signal with a GW model[5] and incoherent non-Gaussian noise with sine-Gaussian wavelets. It also models the PSD with a combination of cubic splines and Lorentzians. It then runs a trans-dimensional Reversible-Jump MCMC to infer a posterior distribution over the signal, glitch and PSD. Finally, one takes a fair draw from the inferred glitch distribution and subtracts it from the detector strain. We can then run inference on the glitch subtracted strain data. Often in LVK analyses, only one glitch draw from the posterior is used (and we indeed follow this standard for most events analyzed). However, it is more accurate to draw several glitches from the glitch posterior and marginalize over them. This is in effect what is done in Ref. [205] where they take three glitch draws, labelled "A", "B", and "C" and run inference on each mitigated frame. For this paper we utilize the same glitch draws as Ref. [205].

We run the analysis on both mitigation techniques for completeness[6]. We also run the analysis using only LIGO-Hanford or only LIGO-Livingston. While there is a positive Bayes factor for eccentricity regardless of which detector combination is used, the evidence for eccentricity is dominated by LIGO Livingston (see Appendix A). We default to reporting results of GW200129 using `gwsubtract` glitch subtracted data, as this is what is used by official LVK analyses of this event. However, we explicitly indicate in the captions which glitch mitigation is being applied. There is very strong evidence for eccentricity in the `gwsubtract` case ($\log_{10}\mathcal{B}_{\rm EAS/QCAS} > 3.5$), but we cannot rule out the possibility that `gwsubtract` under-subtracts the glitch or `BayesWave` over-subtracts the glitch. Since the conclusive evidence of eccentricity is contingent on the systematic uncertainty of glitch mitigation, we do not claim this event to be a bona-fide eccentric event. However, we note that greater than zero evidence for eccentricity is present irrespective of the glitch mitigation (see Figs. 3, 2 and Table I).

For visualization we display the 90% highest density interval of the projected waveforms of GW200129 in LIGO-Hanford and LIGO Livingston in Fig. 4. We use the posterior

[5] Notably, the quasi-circular aligned-spin `IMRPhenomD` [211, 212] is used, but more generally one should use an eccentric waveform model. However, this is computationally expensive (see Ref. [208]).

[6] We also experimented with truncating the glitch by setting the $f_{\rm min} = 50$Hz. However, we note that the majority of waveform difference from eccentricity with GW200129-like parameters is contained in the $20-50$Hz regime so this type of test is inconclusive.

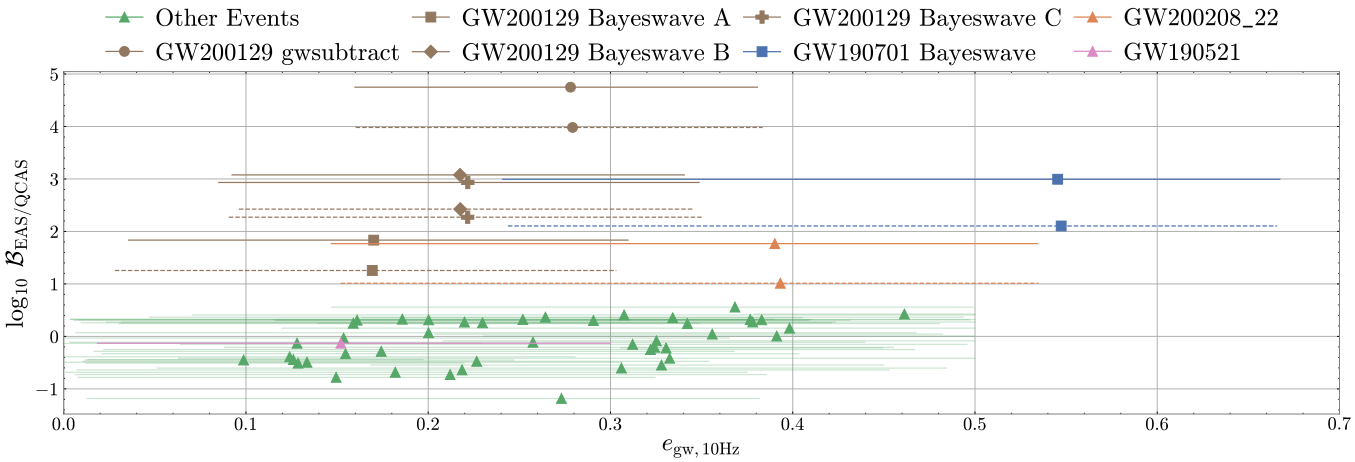


Figure 2. The $\log_{10}\mathcal{B}_{\rm EAS/QCAS}$ and 90% highest-density intervals on $e_{\rm gw,\,10Hz}$ for the 57 events analyzed. Many events have $\log_{10}\mathcal{B}_{\rm EAS/QCAS} < 1$ and we color these green and reduce their opacity. However, there are 3 events GW200129, GW190701 and GW200208_22 which have $\log_{10}\mathcal{B}_{\rm EAS/QCAS} > 1$ which are labelled in the legend. We compute $e_{\rm gw,\,10Hz}$ only for these three events since for events with low support for eccentricity, $e_{\rm gw,\,10Hz} \approx e_{\rm 10Hz}$ [132]. We also color in pink GW190521 which has shown signs of eccentricity in previous papers, but not in our analysis and the one in Ref. [24]. For GW190521, we report the eccentricity at 5.5Hz. The different symbol shapes indicate variations on the glitch mitigation algorithm employed in the analysis (see text). The solid lines indicate a uniform prior on $e_{\rm 10Hz}$ from $[0, 0.5]$ while the dashed lines indicate a log uniform prior on $e_{\rm 10Hz}$ from $[10^{-4}, 0.5]$.

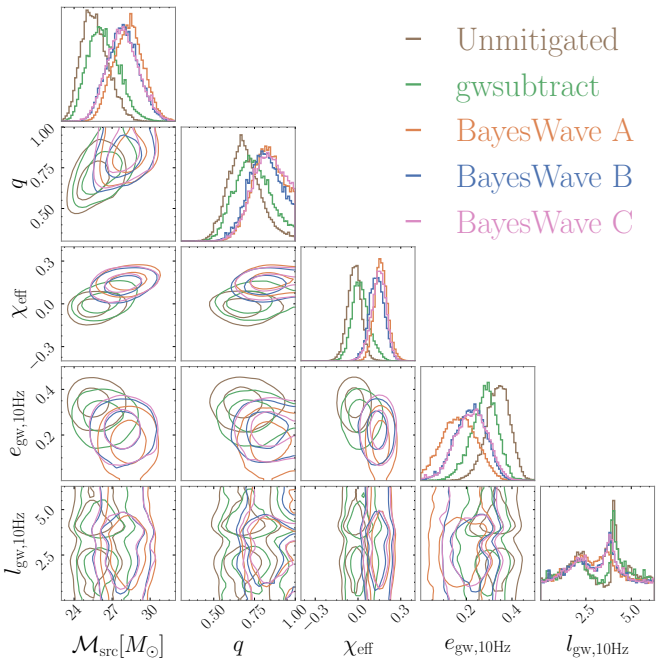


Figure 3. Posteriors of GW200129 using different glitch mitigation techniques. In green is the analysis with `gwsubtract`, which uses auxillary detector channels and transfer functions to subtract glitches. Shown in orange, blue and pink is the analysis carried out using `BayesWave` de-glitching. This method involves generating a posterior of glitches using and then drawing individual glitches from this "glitch posterior". The A, B, and C indicate different fair draws from the glitch distribution and are publicly available at [202].

obtained with the `BayesWave C` glitch mitigation. We also show the location of the `BayesWave C` glitch. To generate this figure, we draw samples from the 90% highest density interval of the `SEOBNRv4EHM` and `SEOBNRv4HM` posteriors, and project them onto the detector. We then whiten the waveform, glitch and strain. By whitening we mean that we set the noise variance in each frequency bin to one using the PSD (see Ref. [213] for further details). We plot the strain data with a rolling window average with a window size of 1 millisecond for visualization purposes. In Fig. 5 we show the radial separation as a function of the coordinate angle of a binary with parameters corresponding to the maximum likelihood of the `SEOBNRv4EHM` analysis of GW200129 with `BayesWave C` glitch mitigation.

Due to the possibility that GW200129 is precessing we additionally compute the Bayes factor against the quasi-circular precessing-spin case. This test is also important as for short signals, it is possible that spin-precession can mask the effect of eccentricity [123, 124]. We first compute the Bayes factors against `SEOBNRv4PHM` and find that the $\log_{10}\mathcal{B}_{\rm EAS/QCP}$ lies in the range $2.20-4.92$. We also compute the Bayes factor against `NRSur7dq4` and find $\log_{10}\mathcal{B}_{\rm EAS/QCNRP}$ in the range $1.43-4.0$. When using the `gwsubtract` glitch mitigation, we find the difference between the maximum $\log_{10}$ likelihoods of `SEOBNRv4EHM` and `NRSur7dq4` is 5.7.

The range of Bayes factors is again due to using different glitch mitigation algorithms. One needs to interpret the $\log_{10}\mathcal{B}_{\rm EAS/QCNRP}$ with caution since eccentricity and precession are not the only differences between `SEOBRv4EHM` and `NRSur7dq4`. In particular, the underlying quasi-circular model of `SEOBNRv4EHM` is different from `NRSur7dq4`. Nonetheless, there is evidence of eccentricity irrespective of the spin-precessing waveform model used. De-

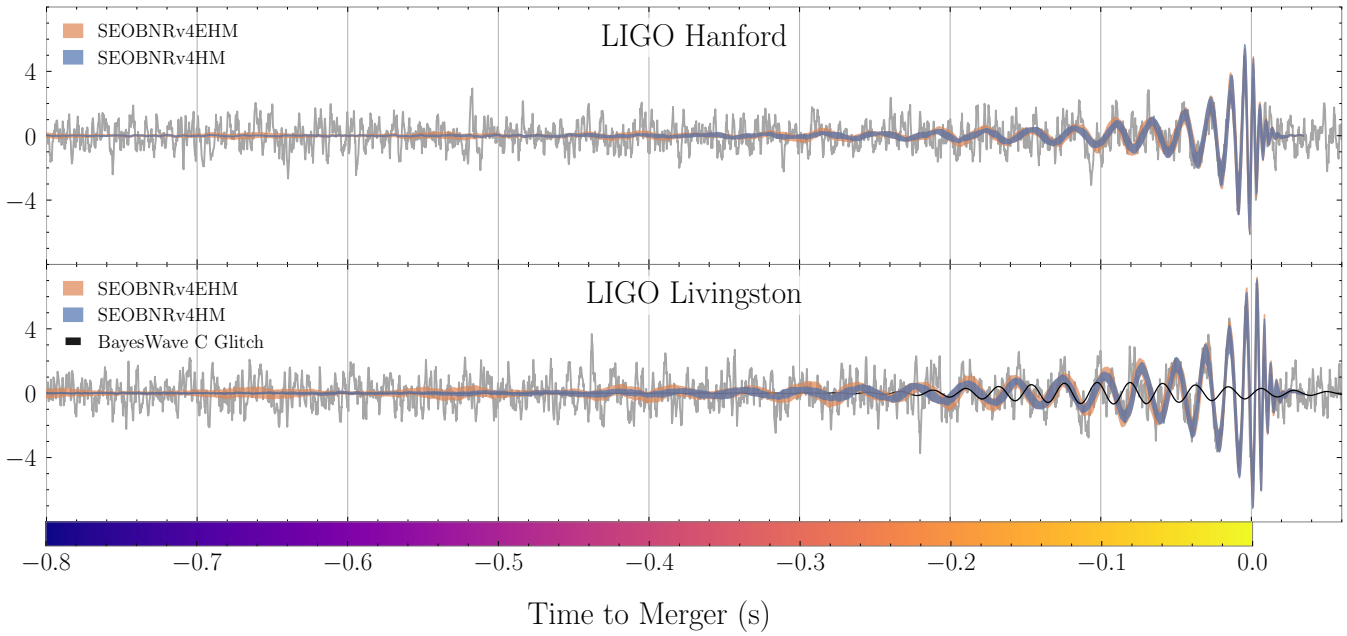


Figure 4. The 90% credible interval of the projected waveforms of GW200129 analyzed with `SEOBNRv4EHM` (orange) and `SEOBNRv4HM` (blue) in LIGO-Hanford (top) and LIGO-Livingston (bottom). The posterior was obtained using `BayesWave C` mitigated data. Shown in light gray is the whitened strain data without any glitch mitigation. We also over-plot the location of the `BayesWave C` glitch in the Livingston detector in black. On the bottom of the plot, we show a color bar that corresponds to the trajectory represented in Fig. 5. The regions in the inspiral where the `SEOBNRv4EHM` waveform deviates from `SEOBNRv4HM` waveform are due to periastron passages of the BBH.

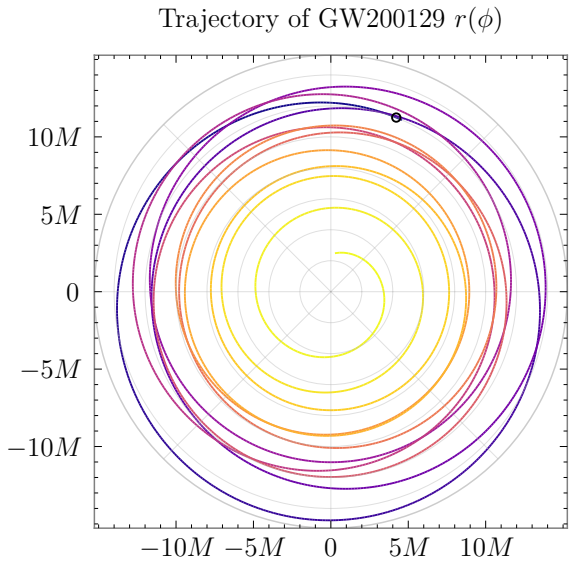


Figure 5. Radial separation as a function of the coordinate angle of a binary with parameters corresponding to the maximum likelihood of the `SEOBNRv4EHM` analysis. The color of the orbit corresponds to the time to merger, and can be compared with the color bar in Fig. 4. The separation is reported in units of the total mass of the system $M$. We mark the start of the orbit with a circle, which is chosen to be $t = -0.8$ seconds before merger.

spite the eccentric aligned-spin hypothesis being preferred over the quasi-circular spin-precessing hypothesis, we do not exclude the possibility that GW200129 is spin-precessing (it could be both eccentric and spin-precessing).

As another check if spin-precession can mimic the eccentricity in this event, we also perform injections of the maximum likelihood waveform of `NRSur7dq4` and recover with `SEOBNRv4EHM` (and vice versa). We conclude that with these parameters, injected precession cannot mimic the effect of eccentricity. Additionally, injected eccentricity does not mimic the effect of precession. The result of this systematics study can be seen in Fig. 13 in Appendix D.

Since `SEOBNRv4EHM` is preferred over `NRSur7dq4` this has important implications for the kick velocity of this event. In particular, Ref. [203] showed evidence for a measurable kick velocity in GW200129. However, when using an eccentric aligned-spin model the evidence for a measurable kick velocity substantially decreases (see Fig. 12 and the discussion in Appendix C).

### D. GW190701

We also see signs for eccentricity in GW190701. To our knowledge GW190701 has not shown signs of eccentricity in previous studies. When analyzed with `SEOBNRv4EHM`, GW190701 is a heavy BBH with $M_{\rm src} = 183.5^{+40.7}_{-39.1}$, $M_{\rm det} = 131.9^{+18.0}_{-17.7}$ and $\chi_{\rm eff} = -0.04^{+0.21}_{-0.26}$. We find that $e_{\rm gw,\,10Hz} =$

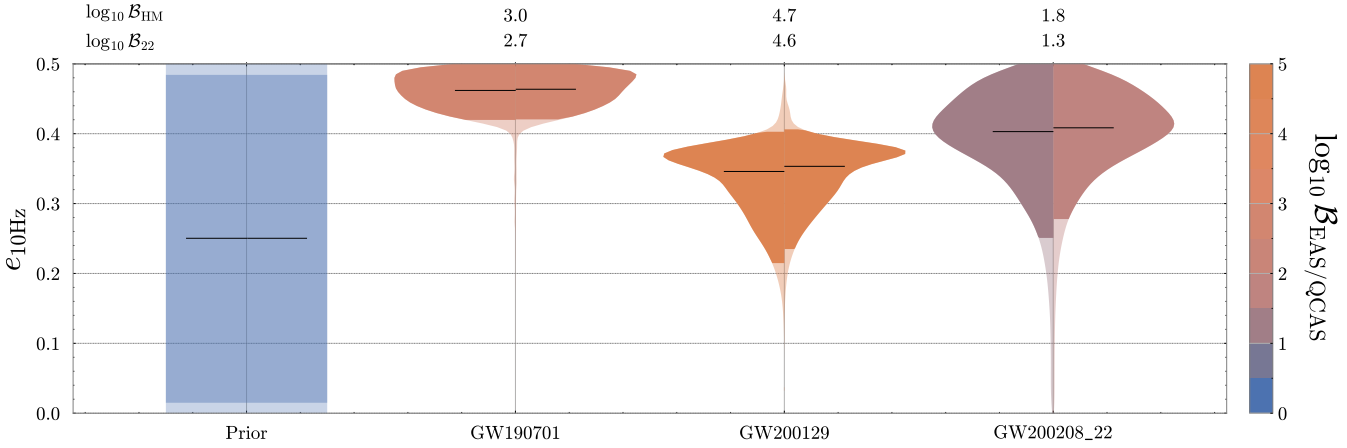


Figure 6. Eccentric violin plots for all events with $\log_{10}\mathcal{B}_{\rm EAS/QCAS} > 1$ from the first through third observing runs of the LVK. The posterior distributions are obtained with `DINGO` and then importance sampled. The eccentricity is measured at 10 Hz in the detector frame using the quasi-Keplerian parameterization (see Eq. (3.1)). For each event, the left violins are the posterior distributions using only the $(\ell,|m|) = (2,2)$ mode, and the right violins are the posteriors using higher modes (see Eq. (3.2)). Each violin is colored according to the $\log_{10}$ Bayes factor between the eccentric aligned-spin and quasi-circular aligned-spin hypothesis. Above the plot, we report the $\log_{10}$ Bayes factor when using higher modes ($\log_{10}\mathcal{B}_{\rm HM}$) and when only using the $(\ell,m) = (2,2)$ mode ($\log_{10}\mathcal{B}_{22}$). The dark shaded regions indicate the 90% credible interval of the distribution and the black lines indicate the median value of the eccentricity.

$0.35^{+0.32}_{-0.11}$ with Bayes factors of $\log_{10}\mathcal{B}_{\rm EAS/QCAS} = 3.0$ and $\log_{10}\mathcal{B}_{\rm EAS/QCP} = 2.61$ (Table I). We find the difference between the maximum $\log_{10}$ likelihoods of `SEOBNRv4EHM` and `SEOBNRv4PHM` is 5.2. Critically, the eccentricity rails against the upper bound of the prior (see Fig. 6). Thus this Bayes factor is a conservative lower bound since there is likelihood support above $e_{\rm 10Hz} > 0.5$. Like GW200129, this event also contains a glitch, which is subtracted using one fair draw of a `BayesWave` glitch model. However, we do not have access to the glitch distribution in this case so we perform the analysis using only one fair draw of the glitch posterior.

We tried to train a `DINGO` network in the regime $e_{\rm 10Hz} < 0.8$, to investigate the high eccentricity of GW190701. However, the gradients of the network increase dramatically in the first 5 epochs. This indicates that the additional hyper-parameters of the network, such as the learning rate need to be further modified. Another possibility is that artifacts in the waveform due to high eccentricity and spins [119] affect substantially the learning process. These issues could be potentially addressed in the future with improved waveform models [214] and network architectures [116].

An important point of caution is that the maximum likelihood waveform of GW190701 has only 3 cycles in band due to the high value of $e_{\rm 10Hz}$. This means we need to back evolve the waveform in order to have enough cycles to compute $e_{\rm gw,\,10Hz}$. However, this means going to values where $e_{\rm 10Hz} > 0.5$ where the waveform model has not been tested against NR. Thus, one needs to interpret the $e_{\rm gw,\,10Hz}$ for this event with some caution. With waveform models which probe and and tested against higher initial eccentricities, this issue can be mitigated [214].

### E. GW200208_22

We also see signs for eccentricity in GW200208_22. The evidence for eccentricity in GW200208_22 has been seen previously in Refs. [20, 23]. When analyzed with `SEOBNRv4EHM`, we observe GW200208_22 is an intermediate-mass BBH with support for positive effective spin ($M_{\rm src} = 94.7^{+28.9}_{-12.5}$, $M_{\rm det} = 66.3^{+17.5}_{-12.5}$ and $\chi_{\rm eff} = 0.12^{+0.30}_{-0.29}$). We find that $e_{\rm gw,\,10Hz} = 0.35^{+0.18}_{-0.21}$, $\log_{10}\mathcal{B}_{\rm EAS/QCAS} = 1.77$ and $\log_{10}\mathcal{B}_{\rm EAS/QCAS} = 1.23$. We find the difference between the maximum $\log_{10}$ likelihoods of `SEOBNRv4EHM` and `SEOBNRv4PHM` is 3.1.

We identify railing of the mass ratio on the lower bound of $q = 0.125$ with `SEOBNRv4HM`, `SEOBNRv4PHM` and `SEOBNRv4EHM`. However, due to the lower bound mass restriction set by `DINGO`, we do not relax this lower bound. Instead, we perform a `pBilby` run extending the lower bound on $q$ to 0.05 to explore the possibility of a secondary mode at low mass ratios. We do not see evidence for such a mode. We also see railing of eccentricity against the eccentricity upper bound of $e_{\rm 10Hz} = 0.5$. But due to the model restriction of `SEOBNRv4EHM` (i.e., $e < 0.5$), we do not increase this upper bound. Like in the case of GW190701 $\log_{10}\mathcal{B}_{\rm EAS/QCAS}$ are then only conservative lower bounds of the Bayes factor.

### F. GW190521

We also train specialized `SEOBNRv4EHM` and `SEOBNRv4HM` networks to analyze GW190521. This event is interesting as it only has 4 cycles in the detectors' bandwidth, and the SNR is dominated by the merger and ringdown. It has been suggested that this event is a head-on collision with exotic compact ob-

| Glitch Subtraction | $e$ Prior | SEOBNRv4 $\log_{10}\mathcal{B}$ | SEOBNRv4HM $\log_{10}\mathcal{B}$ | SEOBNRv4PHM $\log_{10}\mathcal{B}$ | NRSur7dq4 $\log_{10}\mathcal{B}$ | $e_{10\mathrm{Hz}}$ | $e_{\mathrm{gw,\,10Hz}}$ |
|---|---|---|---|---|---|---|---|
| | | | GW200129 | | | | |
| gwsubtract | Uniform | 4.57 | 4.75 | 4.92 | 4.0 | $0.34^{+0.11}_{-0.06}$ | $0.27^{+0.10}_{-0.12}$ |
| BayesWave A | Uniform | 1.7 | 1.84 | 2.20 | 1.53 | $0.24^{+0.10}_{-0.10}$ | $0.17^{+0.14}_{-0.13}$ |
| BayesWave B | Uniform | 2.92 | 3.08 | 3.43 | 2.35 | $0.28^{+0.09}_{-0.11}$ | $0.22^{+0.12}_{-0.13}$ |
| BayesWave C | Uniform | 2.85 | 2.93 | 2.63 | 1.43 | $0.27^{+0.09}_{-0.10}$ | $0.22^{+0.13}_{-0.14}$ |
| gwsubtract | Log-Uniform | 4.02 | 3.98 | 3.99 | 3.23 | $0.33^{+0.07}_{-0.11}$ | $0.28^{0.11}_{-0.12}$ |
| BayesWave A | Log-Uniform | 1.79 | 1.26 | 1.61 | 0.94 | $0.22^{+0.13}_{-0.15}$ | $0.17^{0.14}_{-0.13}$ |
| BayesWave B | Log-Uniform | 2.28 | 2.43 | 2.78 | 1.70 | $0.27^{+0.09}_{-0.11}$ | $0.22^{0.13}_{-0.12}$ |
| BayesWave C | Log-Uniform | 2.22 | 2.27 | 1.97 | 0.76 | $0.26^{+0.10}_{-0.11}$ | $0.22^{0.13}_{-0.13}$ |
| | | | GW190701 | | | | |
| BayesWave | Uniform | 2.72 | 3.0 | 2.61 | – | $0.46^{+0.04}_{-0.04}$ | $0.35^{+0.32}_{-0.11}$ |
| BayesWave | Log-Uniform | 1.86 | 2.11 | 1.71 | – | $0.45^{+0.05}_{-0.04}$ | $0.34^{+0.32}_{-0.10}$ |
| | | | GW200208_22 | | | | |
| Unmitigated Strain | Uniform | 1.25 | 1.77 | 1.23 | – | $0.4^{+0.08}_{-0.15}$ | $0.35^{+0.18}_{-0.21}$ |
| Unmitigated Strain | Log-Uniform | 0.54 | 1.05 | 0.48 | – | $0.35^{+0.15}_{-0.33}$ | $0.35^{+0.19}_{-0.19}$ |

Table I. Bayes factors of the three GW events with $\log_{10}\mathcal{B}_{\mathrm{EAS/QCAS}} > 1$. The first column indicates which glitch mitigation algorithm is used (see text). The second column indicates whether the prior on $e_{10\mathrm{Hz}}$ is is uniform between [0.0, 0.5] or log-uniform between $[10^{-4}, 0.5]$. The third column indicates the Bayes factors between the eccentric aligned-spin $(\ell, m) = (2,2)$ mode only model (SEOBNRV4E) and quasi-circular aligned-spin $(\ell, m) = (2,2)$ mode only model (SEOBNRv4). The fourth through sixth columns indicate the Bayes factors between the eccentric aligned-spin model (SEOBNRv4EHM) against the quasi-circular aligned-spin (SEOBNRv4HM) or quasi-circular precessing-spin (SEOBNRv4PHM or NRSur7dq4) models. The last two columns indicate the mean and 90% highest density interval of the SEOBNRv4EHM posterior for $e_{10\mathrm{Hz}}$ and $e_{\mathrm{gw,\,10Hz}}$ respectively. The errors in each $\log_{10}$ Bayes factor are less than 0.04. For entries with dashes, we do not compute Bayes factors due to a lack of networks that cover the appropriate prior.

jects [215], a non-spinning hyperbolic capture [113], a merger within an active galactic nucleus (AGN) [90, 216, 217] and eccentric [21, 110]. It has also been suggested that this event has an electromagnetic counterpart detected by the Zwicky Transient Facility (ZTF) [216].

In order to analyze GW190521, we train a specialized network with a larger upper bound on the detector frame component masses ($m_{1,2} < 180 M_\odot$). The network also has a starting frequency of $f_{\mathrm{start}} = 5.5$ Hz. This is to make sure the higher modes, and in particular the (4,4) mode, are in band at the minimum frequency, which we set to $f_{\mathrm{min}} = 11$ Hz. Thus the eccentricity is sampled at 5.5 Hz ($e_{5.5\mathrm{Hz}}$). Accordingly, we adjust the prior in the eccentricity to be uniform between (0, 0.3).

When using SEOBNRv4EHM, we recover $e_{\mathrm{gw,\,5.5Hz}} = 0.12^{+0.12}_{-0.12}$ for GW190521 with a posterior very close to the prior. We find a $\log_{10}\mathcal{B}_{\mathrm{EAS/QCAS}}$ of 0.04. This is likely due to the same reasons described in Ref. [24]. Notably, GW190521 is a merger-ringdown dominated signal, but the eccentric SEOBNRv4EHM waveform is the same as the quasi-circular SEOBNRv4HM waveform during the merger-ringdown. Thus it is difficult to measure eccentricity. So while we do not see signs of eccentricity in this analysis, the high eccentricity limit near merger is fairly unconstrained. We highlight that this is a common feature for all current eccentric waveform models in the literature.

Finally, to ensure the analysis of GW200129, GW190701, GW200208_22 and GW190521 do not depend on the sampler, we perform additional pBilby analyses using SEOBNRv4E (SEOBNRv4EHM but turning off the higher modes) of these

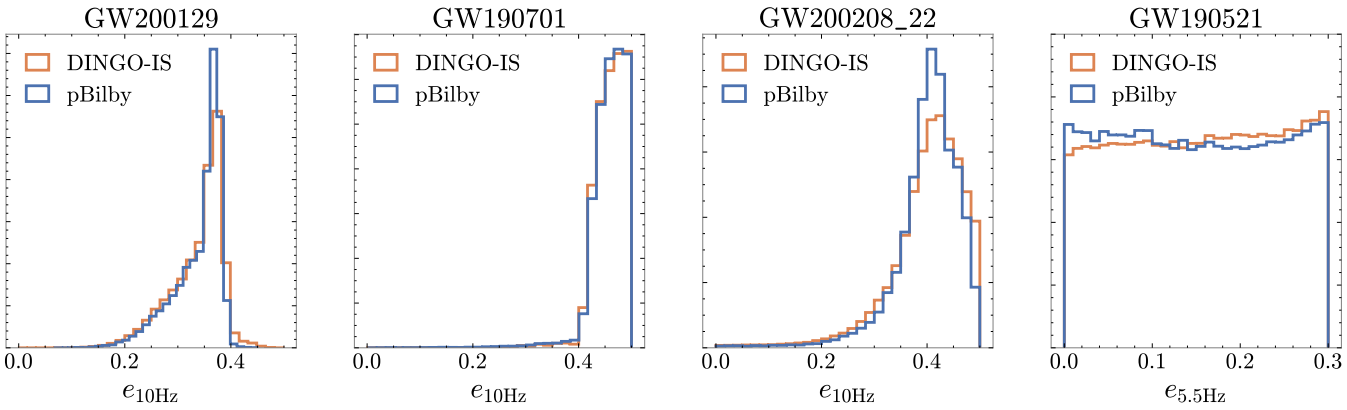


Figure 7. `DINGO-IS` versus `pBilby` marginal eccentricity distributions for four of the events analyzed in the main text. We show that the presence of eccentricity is also found by nested samplers. Note that while, there are are slight differences between `DINGO-IS` and `pBilby` in the GW200208_22 posterior, this can be attributed to the railing of the eccentricity distribution at 0.5. When extending the prior of the `pBilby` run, we again find better agreement.

events in Fig. 7.

### G. Events with mild eccentricity support

We find marginal evidence for eccentricity in GW190620_030421. When analyzed with `SEOBNRv4EHM`, we observe this is a heavy BBH with support for a positive effective spin with $M_{\mathrm{src}} = 168.3^{+36.4}_{-34.9}$, $M_{\mathrm{det}} = 126.6^{+18.7}_{-17.9}$ and $\chi_{\mathrm{eff}} = 0.20^{+0.26}_{-0.27}$. Unlike the previously discussed events in this paper, GW190620_030421 was only detected in LIGO Livingston. We find that $\log_{10}\mathcal{B}_{\mathrm{EAS/QCAS}} = 0.56$ with $e_{\mathrm{gw,\,10Hz}} = 0.30^{+0.19}_{-0.19}$. While this event does have 97% of it's posterior above $e_{\mathrm{gw,\,10Hz}} > 0.05$, due to the low Bayes factor, we do not consider it as having evidence for eccentricity. This event has shown similar levels of eccentricity in Refs. [20, 23].

We also analyze GW191109_010717, which has shown signs of eccentricity in Ref. [20]. When analyzed with `SEOBNRv4EHM`, we find this is a heavy BBH with $M_{\mathrm{src}} = 175.6^{+34.0}_{-30.0}$, $M_{\mathrm{det}} = 137^{+14.0}_{-14.4}$ and $\chi_{\mathrm{eff}} = -0.25^{+0.24}_{-0.27}$. While we do notice that 93% of the posterior has support above $e_{10Hz} > 0.05$, we also find $\log_{10}\mathcal{B}_{\mathrm{EAS/QCAS}} = 0.07$ with $e_{\mathrm{gw,\,10Hz}} = 0.26^{+0.24}_{-0.22}$ for this event. Interestingly, the posterior does rail against the upper bound of the prior, but due to the eccentricity limit enforced by `SEOBNRv4EHM`, we do not increase the upper bound. This event also has two peaks in the eccentricity posterior with one at $e_{\mathrm{gw,\,10Hz}} = 0.27$ and the other and the edge of the prior $e_{\mathrm{gw,\,10Hz}} = 0.50$. This was also noted in Ref. [20].

## V. WAVEFORM SYSTEMATICS

### A. Biases from neglecting eccentricity

When inferring source parameters with a waveform model that does not include the effect of eccentricity, one can bias the inferred source parameters. This effect is particularly pronounced in the three events that show signs for eccentricity where biases in the chirp mass and effective spin are observed. The results of this analysis are shown in Fig. 8. One of the main effects is that the eccentricity reduces the time to merger. To compensate for this, other parameters must change keeping the true time to merger fixed. In the cases of GW200129 and GW200208_22, the chirp mass decreases to keep a fixed time to merger. We observe chirp-mass differences between `SEOBNRv4EHM` and `SEOBNRv4HM` of 5.2% ($-1.54M_{\odot}$) for GW200129, 3.6% ($2.00M_{\odot}$) for GW190701 and 11.8% ($-3.08M_{\odot}$) GW200208_22. We observe chirp-mass differences between `SEOBNRv4EHM` and `SEOBNRv4PHM` of 5.2% ($-1.56M_{\odot}$), 2.4% ($1.32M_{\odot}$), and 11.6% ($-3.03M_{\odot}$) in GW200208_22

There is also a bias in $\chi_{\mathrm{eff}}$ across the events. We observe differences of $-0.04$, 0.12 and $-0.19$ in the mean $\chi_{\mathrm{eff}}$ for GW200129, GW190701 and GW200208_22 respectively. In conclusion, when including the effect of eccentricity in these events, the $\chi_{\mathrm{eff}}$ distribution tends towards 0.

### B. Biases from neglecting higher modes in eccentric waveforms

We investigate the impact of neglecting higher modes beyond the $(\ell, |m|) = (2, 2)$ mode in the estimation of source parameters. It has been shown that neglecting them in quasi-circular models can lead to biases in parameter estimation [2, 218–222]. Here we extend the study to eccentric waveforms. We quantify this by calculating the JSD between

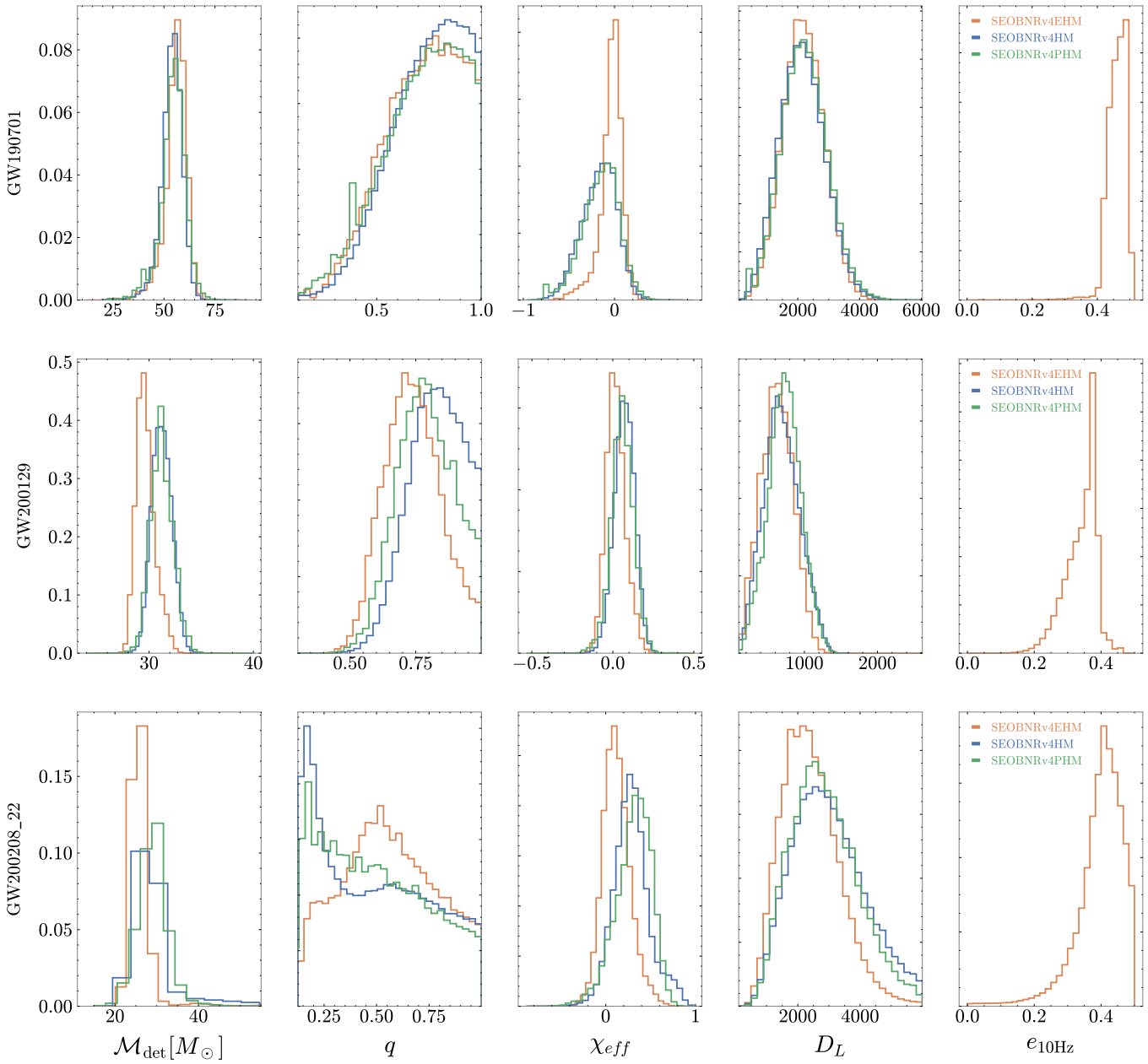


Figure 8. Posterior distributions for the 3 candidate eccentric events analyzed with `SEOBNRv4EHM`, `SEOBNRv4PHM`, and `SEOBNRv4HM`. The posteriors were obtained by DINGO and then importance sampled (see II C). When ignoring the effects of eccentricity, we see large differences in the $\mathcal{M}_{\rm det}$ and $\chi_{\rm eff}$. Noting that eccentricity decreases the time to merger with respect to quasi-circular templates, the biases can be understood as compensations by $\mathcal{M}_{\rm det}$ and $\chi_{\rm eff}$ to keep the time to merger fixed.

the analysis performed with `SEOBNRv4E` and `SEOBNRv4EHM`. For comparison we also compute the JSDs for `SEOBNRv4` and `SEOBNRv4HM`. We follow Ref. [2] and consider deviations beyond 0.007 bits as significant biases. For a Gaussian, this corresponds to a 20% shift in the mean measured in standard deviations of the Gaussian. (We remark that 0.002 bits is the expected stochastic variation from GW sampling algorithms [162]).

First, we find that there are biases in the eccentricity distribution when not including higher modes. This is especially true for GW191109_010717, which can be seen in Fig. 9. In particular, we see that $e_{10\rm Hz}$ is boosted to higher values and leads to a JSD of 0.025 bits when using only the $(\ell,|m|)=(2,2)$ mode. To rephrase, neglecting eccentricity results in increasing the mean eccentricity by 0.034. For this event neglecting higher modes also leads to larger Bayes factors. When only using the $(\ell,|m|)=(2,2)$ mode, $\log_{10}\mathcal{B}^{\rm EAS/QCAS}=0.32$ but when using higher modes $\log_{10}\mathcal{B}^{\rm EAS/QCAS}=0.07$.

Furthermore, neglecting higher modes in eccentric waveforms can lead to biases in source parameters other than the eccentricity. For example, the inclusion of higher modes in eccentric waveforms for GW190701 leads to a bias (0.03 bits) in the mass ratio. In contrast, when neglecting higher modes in the quasi-circular waveforms (that is comparing `SEOBNRv4` to `SEOBNRv4HM`), we see lower difference (0.01 bits). This is also true of the $\chi_{\rm eff}$ distribution of GW200208_22 where neglecting higher modes in eccentric waveforms leads to a 0.01 bit difference whereas neglecting higher modes in quasi-circular waveforms only leads to a 0.002 bit difference.

## VI. ODDS RATIOS AND $p_{\rm ecc}$

### A. Prior odds and odds ratio

We now turn our attention to computing the odds ratio in Eq. (2.4) by taking into account the effect of astrophysical prior odds on our observations. To do this we first put a prior on the rate of eccentric BBHs, $R_{\rm E}$, and quasi-circular BBHs, $R_{\rm QC}$, that we expect. We then take the maximum probability of this prior to estimate the prior odds.

We define $R_{\rm E}$ as the number of events with $e_{\rm 10Hz} > 0.05$ per year within a luminosity distance of 6000 Mpc ($z < 0.9$) [96]. The reason we select $z < 0.9$ is because this is the maximum distance we consider in the analysis. Similarly, we define $R_{\rm QC}$ as the number of events with $e_{\rm 10Hz} < 0.05$, again within $z < 0.9$.

The first step is to place a prior on the merger rate of eccentric and quasi-circular events based on theoretical predictions[7]. We consider four binary formation environments that may lead to BBHs. The principal difference between these formation environments is the cluster escape speed, which increases the number of dynamical encounters [68]. We consider nuclear-star clusters (NSCs), globular clusters (GCs), young-star clusters (YSCs) and isolated binaries (IBs). We assume that each channel is independent and contributes some rate of eccentric events, $R^{\rm c}_{\rm E}$, and some rate of quasi-circular events, $R^{\rm c}_{\rm QC}$. The superscript $c$ denotes the name of the channel. Each of these channels contribute to the total rate of eccentric or quasi-circular events:

$$\begin{aligned} R_{\rm E} &= R^{\rm GC}_{\rm E} + R^{\rm NSC}_{\rm E} + R^{\rm YSC}_{\rm E} + R^{\rm IB}_{\rm E} \\ R_{\rm QC} &= R^{\rm GC}_{\rm QC} + R^{\rm NSC}_{\rm QC} + R^{\rm YSC}_{\rm QC} + R^{\rm IB}_{\rm QC}. \end{aligned} \tag{6.1}$$

We can then write the prior over the total rates as a convolution over the channel rates.

$$\begin{aligned} p(R_{\rm E}, R_{\rm QC}) = p(R^{\rm GC}_{\rm E}, R^{\rm GC}_{\rm QC}) * p(R^{\rm NSC}_{\rm E}, R^{\rm NSC}_{\rm QC}) \\ * p(R^{\rm ysc}_{\rm E}, R^{\rm ysc}_{\rm QC}) * p(R^{\rm IB}_{\rm E}, R^{\rm IB}_{\rm QC}). \end{aligned} \tag{6.2}$$

In the literature, the total rates from each channel (denoted $R^c$) and the fraction of the total rate that have $e_{\rm 10Hz} > 0.05$ (denoted $\gamma^c$) are reported [88, 91, 223, 224]. Thus, we place a prior on $R^c$ and $\gamma^c$ and do a change of variables to obtain $p(R^c_{\rm E}, R^c_{\rm QC})$. Explicitly, this change of variables is:

$$p(R^c_E, R^c_{QC}) = p(R^c, \gamma^c) \left[ \frac{1}{\gamma^c R^c} - \frac{1}{(1-\gamma^c) R^c} \right], \tag{6.3}$$

where the term in the brackets is the Jacobian transformation between $(R^c_{\rm E}, R^c_{\rm QC})$ and $(R^c, \gamma^c)$.

We first discuss the priors on the fractional rates, $\gamma^c$. NSCs can host BBH mergers due to the high density ($2\times10^4\,{\rm pc}^{-1}$) of BHs [70–73, 77, 89, 225–228]. Mergers can also be induced by dynamical friction in the galactic disc [229–235]. The high BH density in NSCs leads to GW captures, which lead to high eccentricities ($e_{\rm 10Hz} > 0.1$) [77, 80, 81, 224, 236–238]. If binary-single BH interactions are constrained to the nuclear disc, 10-70% of mergers have $e_{\rm 10Hz} > 0.1$, whereas if interactions are isotropic, then 8-30% of mergers have $e_{\rm 10Hz} > 0.03$ [89]. Thus we place a uniform prior on $\gamma^{\rm NSC}$ with a minimum of 8% and a maximum of 70%.

GC environments can also host BBH mergers due to the high density of BHs [70–73] Ref. [87] finds $\gamma_{\rm GC} \sim 6\%$ (Ref. [224] finds similar results with a PN calculation). This method includes PN corrections to encounters between BHs and BBHs using techniques developed in Ref. [88, 239, 240], which lead to non-negligible eccentricity. Thus, we place a truncated Gaussian prior around $\gamma^{\rm GC} = 6\%$ with a standard deviation of 3%, which comes from the variation in the cluster escape velocity (see Fig. 3 of Ref. [224]).

Studies on the rate of eccentric mergers in YSCs are less common than NSCs or GCs (see however, Refs. [82] and [223]). Ref. [223] finds less than 0.08% of mergers in YSCs have $e_{\rm 10Hz} > 0.1$ from binary-single interactions. Thus we neglect this effect and set $p(\gamma^{\rm YSC}) = \delta(0)$. Similarly, in IBs, we expect $p(\gamma^{\rm IB}) = \delta(0)$ as the binaries circularize before merger [41, 184].

We now turn to placing a prior over the total rates from each channel, $p(R^c)$. There have been a number of studies estimating the rate density of NSCs [241, 242], GCs [87, 243–245], YSCs [246–248], and IBs [18, 248–250]. In the following, we use the rate estimates from Ref. [251] as they model the IB and NSC, GC and YSC pathways with the same input physics/assumptions. Since, we are interested in the merger rate of BBHs within $z < 0.9$ and not the merger rate density as a function of redshift, which is computed in [251], we may write $R^c$ as

$$R^c = \int_0^{0.9} dz \mathcal{R}(z) \frac{dV}{dz} \frac{1}{1+z}. \tag{6.4}$$

[7] An important caveat is that the $e_{\rm 10Hz}$ reported in astrophysical studies is *not* the same as the $e_{\rm 10Hz}$ in the quasi-Keplerian parameterization (Eq. 3.1) [86]. Since the point which we use to define the rate is $e_{\rm 10Hz} \approx 0.05$ and the discrepancy between the two definitions becomes relevant at $e_{\rm 10Hz} \geq 0.2$, we do not expect this to be a significant issue for defining the heuristic prior.

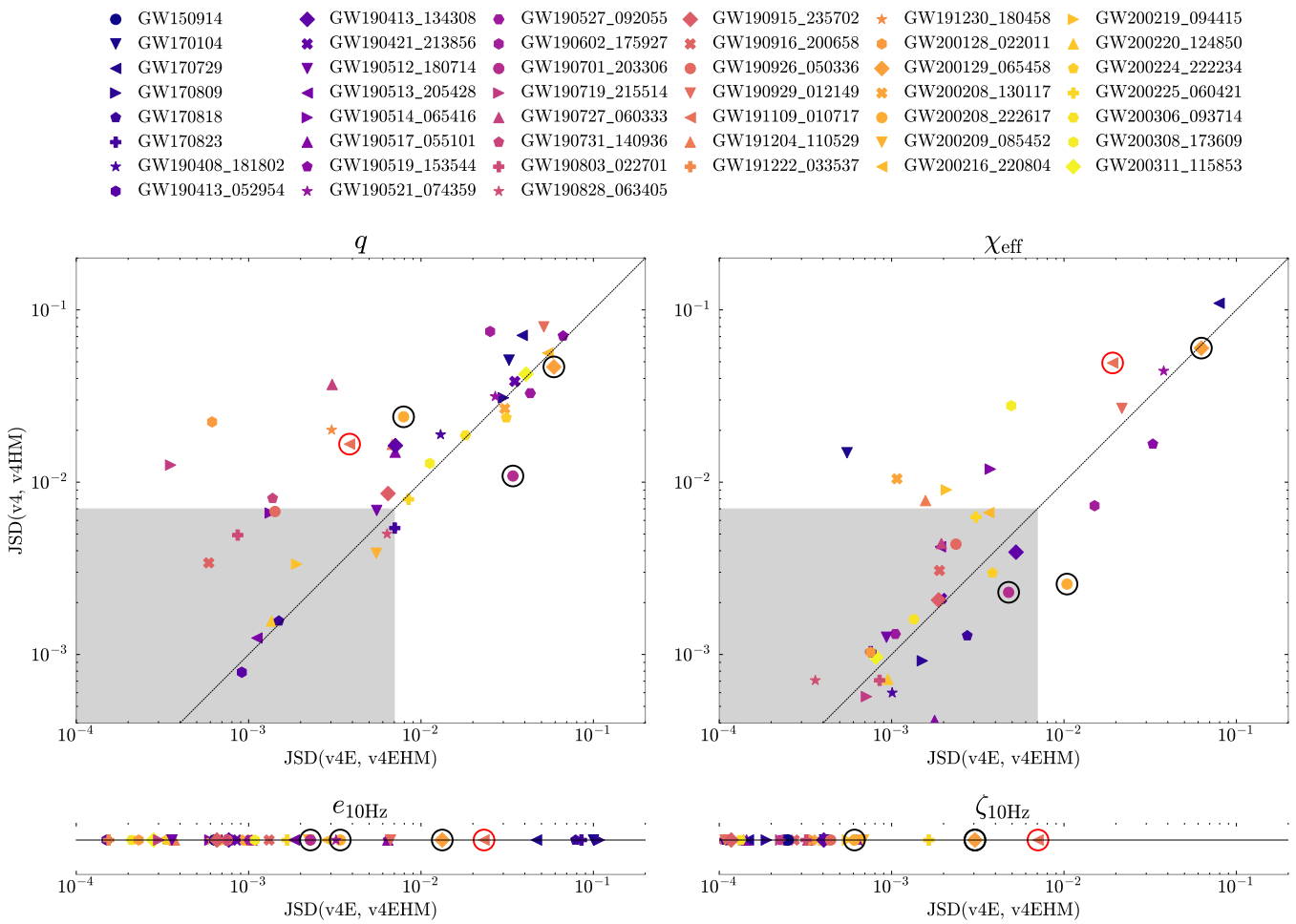


Figure 9. JSDs of 1D marginals when including or not including higher modes in GWs. Each point represents a GW event and contains four (top) or two (bottom) posterior distributions. In the top plot, the $y$-axis shows the JSD between `SEOBNRv4` and `SEOBNRv4HM` in bits. On the top and bottom plots the $x$-axis shows the JSD between `SEOBNRv4E` and `SEOBNRv4EHM` in bits. The title of the plots indicate which 1D marginal is being compared (in order mass ratio, effective spin, eccentricity and relativistic anomaly). The gray section indicates the region in which the JSD is less than 0.007 bits. For a Gaussian, this corresponds to a 20% shift in the mean measured in standard deviations of the Gaussian. Events to the right of the diagonal line represent events for which the eccentric higher-modes are important. For events like GW190701, GW200208_22 and GW200129 (circled in black), which show signs of eccentricity, there are larger JSDs between `SEOBNRv4E` and `SEOBNRv4EHM` than `SEOBNRv4` and `SEOBNRv4HM`. In addition, higher modes are important for the analysis of GW191109_010717 (circled in red) as they cause large deviations in the eccentricity and relativistic anomaly posterior.

Here, $\mathcal{R}(z)$ is the volumetric rate density in units of mergers $\mathrm{Gpc}^{-3}\mathrm{yr}^{-1}$ as a function of redshift and $dV/dz$ is the differential co-moving volume. The $\frac{1}{1+z}$ accounts for the fact that $\mathcal{R}(z)$ is in the source frame whereas we want $R^c$ in the observer frame. We can perform the integration for the set of simulations listed in Ref. [251] and set lower and upper bounds on the rates. Then, we assume that $R^c$ is drawn from uniform distribution between these bounds. The final priors on $\gamma^c, R^c$ are summarized in Table II. The prior can be seen in 1D marginals on the left plot in Fig. 10.

The maximum probability of the prior occurs when $R_{\rm E}/R_{\rm QC} = 0.023$, and thus we quote this value for our prior odds. Now using Eq. (2.4) we obtain

$$\begin{aligned}\log_{10}\mathcal{O}_{\rm EAS/QCAS} &= 0.2-3.11 \quad \text{for GW200129,}\\ &= 1.36 \quad \text{for GW190701,}\\ &= 0.13 \quad \text{for GW200208_22.}\end{aligned}$$

### B. Computing $p_{\rm ecc}$ for three events

We now compute the rate of eccentric and quasi-circular events using GW observations. We then use these rates to estimate the probability an event is eccentric for the three GW events considered in Sec. IV. This is done by using the formal-

| formation environment | $p(\gamma^c)$ | $p(R^c)$ mergers/yr |
|---|---|---|
| nuclear-star cluster | Unif[0.08, 0.7] | Unif[119, 292] |
| globular cluster | $\tilde{\mathcal{N}}(0.06, 0.03)$ | Unif[375, 1492] |
| young-star cluster | $\delta(0)$ | Unif[502, 2325] |
| isolated binary | $\delta(0)$ | Unif[639, 31939] |

Table II. Priors on the fractional rate of eccentric BBH mergers, $\gamma^c$ and total rate of BBH mergers $R^c$ in each formation channel within $z < 0.9$. Here Unif$[a,b]$ is a uniform distribution with upper and lower bound of $a$ and $b$, respectively. $\tilde{\mathcal{N}}[c,d]$ is a truncated normal distribution with mean and standard deviation $c$ and $d$, respectively, with truncation bounds of 0 and 1.

ism in Eqs. (2.5)– (2.16). The rates obtained with this analysis using an astrophysical prior and a uniform prior over the rates is shown in Fig. 10.

We also include selection effects in this estimate. To do so we compute $\alpha_{\rm QC}$ and $\alpha_{\rm E}$ with Eq. (2.15). For the prior, we use a power-law–peak distribution with hyperparameters of the maximum-likelihood point of the LVK analysis [13]. In addition, for any parameters that are outside the prior, ($m_1 < 10M_\odot$, $m_2 < 10M_\odot$, $q < 0.125$, or $d_L > 6{,}000$ Mpc), we set $p_{\rm QC,det}(\boldsymbol{\vartheta}) = p_{\rm E,det}(\boldsymbol{\vartheta}) = 0$. Due to the fact that the selection function also depends on the PSD, we compute $\alpha_{\rm QC}$ and $\alpha_{\rm E}$ for each observing run. We then take a weighted average of the $\alpha$'s where the weights are determined by the joint duty cycle of LIGO Hanford and LIGO Livingston for each observing run (49 days, 118 days, 107 days and 96 days for O1, O2, O3a and O3b, respectively) .

There are a few interesting observations to make about Fig. 10. First, there is a large difference in $p(R_{\rm E}|\{d_i\},N)$ when using an astrophysical prior versus a uniform one. Specifically, the astrophysical prior has a strong influence over the eccentric rate. The reason for this that the uncertainty on the fraction of eccentric BBHs quoted in Sec. VI A is small compared to the uncertainty on the fraction of eccentric BBHs from GWs. The latter uncertainty can be seen by noticing that in Fig. 2 the majority of events have $-1 < \log_{10}\mathcal{B}^{\rm EAS/QCAS} < 1$ as opposed to $\log_{10}\mathcal{B}^{\rm EAS/QCAS} \ll -1$. In other words, if just considering the data, it is hard to say the majority of events are definitely *not* eccentric. With better detector sensitivities, perhaps we can better separate the populations. This is also the reason that when using a uniform prior on the rate of eccentric and quasi-circular BBHs, the mean of $p(R_{\rm E}|\{d_i\},N)$ is only larger than $p(R_{\rm QC}|\{d_i\},N)$ by 14% (361 events/yr).

In contrast, the astrophysical prior does not have the same effect on $p(R_{\rm QC}|\{d_i\},N)$. Here, $p(R_{\rm QC}|\{d_i\},N)$ is dominated by the information from the data. This effect can be attributed to the large uncertainty in the rate of quasi-circular events from astrophysical simulations [251].

We can compute the probability that particular GWs are eccentric, $p_{\rm ecc}$, using Eq. (2.11). Again due to the variety of glitch mitigation techniques in GW200129, we have a range of $p_{\rm ecc}$'s. This uncertainty now also propagates into events other than GW200129 as $p_{\rm ecc}$ for each event is also conditioned on the GW data from GW200129. When using an astrophysical prior on the rates, we obtain

$$\begin{aligned} p_{\rm ecc} &= 0.7490 - 0.9995 \quad \text{for GW200129,} \\ &= 0.9763 - 0.9774 \quad \text{for GW190701,} \\ &= 0.7197 - 0.7286 \quad \text{for GW200208_22.} \end{aligned} \tag{6.5}$$

When using a uniform prior on the rates, we find that

$$\begin{aligned} p_{\rm ecc} &= 0.9800 - 0.99998 \quad \text{for GW200129,} \\ &= 0.9986 - 0.9986 \quad \text{for GW190701,} \\ &= 0.9769 - 0.9771 \quad \text{for GW200208_22.} \end{aligned} \tag{6.6}$$

We can now compute the probability that there exists at least one event which is eccentric in the population. First, we give a back-of-the-envelope computation of $p_{\rm ecc,\,pop}$ using only $\mathcal{O}^{\rm EAS/QCAS}$. We stress that this is *not* a robust estimate as it assumes that the inferred rate of eccentric and quasi-circular BBHs is not influenced by GW observations. In addition, it does not marginalize over the astrophysical uncertainty on the rates. This calculation is meant to give an intuition and sanity check on the machinery in Sec. II B. We can use Eq. (2.12) to map the odds ratios of the analyzed events to the interval $[0,1]$. We can then combine the events by computing

$$\tilde{p}_{\rm ecc,\,pop} = 1 - \prod_{i=1}^{N} [1 - \tilde{p}_{\rm ecc}(i)]\,. \tag{6.7}$$

If we use `BayesWave A` glitch mitigation on GW200129, we recover $\tilde{p}_{\rm ecc,\,pop} = 99.8\%$. We can also perform the same calculation, but only use the three events that show signs of eccentricity. In this case, we find $\tilde{p}_{\rm ecc,\,pop} = 99.3\%$.

Finally, we can use the machinery in Eq. (2.13), and Eqs. (B1)– (B7) to robustly compute the probability that there is at least one eccentric event in the population. When using an astrophysical prior on the rates and `gwsubtract` glitch mitigation on GW200129, we find

$$\begin{aligned} p_{\rm ecc,\,pop} &> 1 - 8.4\times10^{-8} \text{ using } \texttt{gwsubtract}, \\ &> 1 - 4.7\times10^{-3} \text{ using } \texttt{BayesWave A}. \end{aligned} \tag{6.8}$$

On the other hand, if we use a uniform prior on the rates, we find

$$\begin{aligned} p_{\rm ecc,\,pop} &> 1 - 7.6\times10^{-14} \text{ using } \texttt{gwsubtract}, \\ &> 1 - 6.2\times10^{-11} \text{ using } \texttt{BayesWave A}. \end{aligned} \tag{6.9}$$

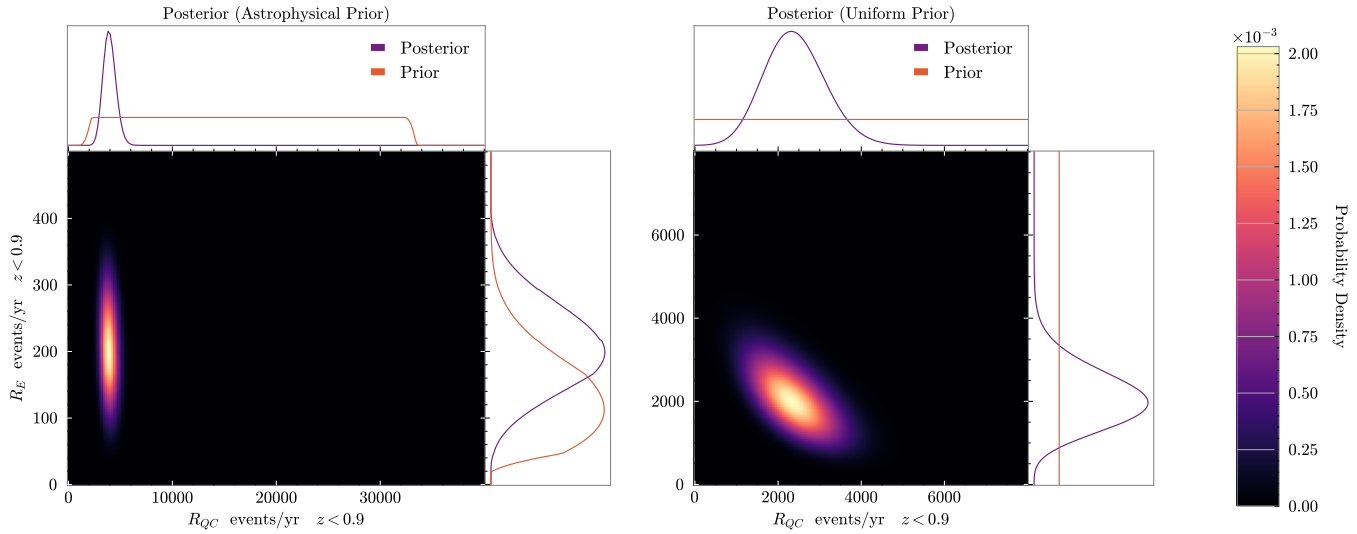


Figure 10. Posterior distribution for the rate of eccentric $R_{\rm E}$ and quasi-circular $R_{\rm QC}$ events per year within $z < 0.9$. This is computed using the formalism of Ref. [131] (see Sec. II B). Shown on the left is the rate estimate combining the astrophysical prior with the GW data. In this case, the prior dominates over the information from the GWs. Shown on the right is the rate distribution estimated from GWs when using a uniform prior for $R_{\rm E}$ and $R_{\rm QC}$. In this case, the quasi-circular and eccentric rates are similar since for the majority of events we cannot distinguish if $e_{10\rm Hz} < 0.05$ or $e_{10\rm Hz} > 0.05$ from the data alone. Note the scales on the plots are different.

## VII. CONCLUSION

In this paper we performed an analysis of 57 GWs using the multipolar eccentric non-precessing waveform model `SEOBNRv4EHM` [133]. This waveform model includes eccentricity corrections to the GW multipoles up to the 2PN order, and allows for sampling in both the eccentricity and relativistic anomaly. This large scale analysis was made possible with the normalizing-flow approach available with `DINGO` [153].

We compare `DINGO` with `pBilby` [134] using two zero-noise injections, and find consistency between the samplers. The posteriors are completely consistent with JSDs in the 1D marginals less than 0.002 bits. This gives us confidence that `DINGO` is able to achieve the same accuracy as standard samplers for eccentric waveforms, while requiring orders of magnitude lower compute.

There are signs of eccentricity in three candidate events: GW200129 GW190701 and GW200208_22. We find $\log_{10}$ quasi-circular aligned-spin against quasi-circular precessing Bayes factors of 1.84-4.75 for GW200129, 3.0 for GW190701 and 1.77 for GW200208_22. While we cannot say conclusively that these three binaries are eccentric, the eccentric hypothesis fits the data better than the quasi-circular aligned-spin and quasi-circular spin-precessing hypotheses. These signs indicate that eccentricity is an integral consideration in future and current parameter-estimation studies.

To our knowledge, this is the first time GW200129 has been fully sampled with an eccentric waveform model. In the Appendix of Ref. [20] the authors tried to sample this event using the re-weighting technique introduced in Ref. [252] with the waveform from Ref. [169]. The re-weighting in Ref. [252] involves taking a quasi-circular proposal distribution and importance sampling to an eccentric target distribution. However, for this event the re-weighting led to an effective sample size of one. This may be the case for events where the quasi-circular proposal is significantly biased by the presence of eccentricity. By directly performing inference on the eccentric parameter, we are able to obtain over 5,000 effective samples across all priors, glitch mitigation techniques and waveform models for this event.

We do see very strong evidence for eccentricity in GW200129 when using `gwsubtract` glitch mitigation. But when taking into account the astrophysical prior odds and other methods of glitch mitigation, we cannot conclude that GW200129 is eccentric. In particular the $\log_{10}$ odds ratio between the eccentric aligned-spin and quasi-circular aligned-spin waveform is between $0.2 - 3.11$. This implies the importance of robust glitch mitigation techniques and thorough comparisons between said techniques is crucial in current and future observing runs [204–206, 209, 210].

We also see signs of eccentricity in GW190701. However, because $\log_{10}$ eccentric aligned-spin against quasi-circular spin-precessing odds ratio for this event is $< 3.5$, we cannot conclude this event is eccentric. This event also has a glitch in it, however, we only have one glitch draw from the `BayesWave` glitch posterior [207]. In future studies, it will be useful to explicitly marginalize over the glitch posterior, as was done in [205]. Given the nature of `DINGO`, marginalizing over different glitch realizations is straightforward.

We also find signs of eccentricity in GW200208_22. This has been reported previously in Ref. [20]. We see mild support for eccentricity in GW190620, which has been flagged as potentially eccentric first in Ref. [20], and then in Ref. [23].

We do not observe evidence of eccentricity in GW190521 in the region $e_{10\rm Hz} < 0.5$. This is in contention with Refs. [21, 110]. However, [23] which uses the `TEOBResumS-DALI`

waveform model [149–151], finds similar results. It is at present unclear what the cause of this discrepancy is, and further work in this direction is needed. Note, our result is not immediately in contention with Ref. [113] which probes the parameter space of non-spinning hyperbolic captures which is separate from eccentric inspiral-merger-ringdown waveforms.

We also perform a study of the effect of higher modes beyond the $(\ell, m) = (2, 2)$ mode in eccentric waveform models. We find that there are biases in the intrinsic parameters when one neglects higher modes in parameter-estimation studies. In the quasi-circular case, this has been found in Refs. [2, 218–222]. Our contribution is to study the biases which occur in the presence of eccentricity. We have quantified this effect by comparing the loss of information accumulated when neglecting higher modes in eccentric waveforms with the loss of information accumulated when neglecting higher modes in quasi-circular waveforms. We find with GW191109_010717 that ignoring higher modes leads to biases in the eccentricity distribution of 0.025 bits. For context, with a Gaussian, an 0.007 bit difference corresponds to a 20% shift in the mean measured in standard deviations of the Gaussian. In addition, if we neglect higher modes in GW191109_010717, the $\log_{10}$ Bayes factor changes from 0.07 to 0.32. We do not find in the events analyzed that neglecting higher modes leads to incorrectly finding evidence for eccentricity.

We also investigated the systematic effects of ignoring eccentricity completely in parameter-estimation studies. Similar studies have been performed in Refs. [23, 24, 92–102]. Our new contribution is to report the bias in the masses and spins for real events with support for $e_{10\mathrm{Hz}} > 0.1$. We find that when ignoring eccentricity, the chirp mass is biased by 5.2% for GW200129, 3.6% for GW190701 and 11.8% GW200208_22 with respect to the aligned-spin quasi-circular case. Similarly, the $\chi_{\mathrm{eff}}$ is biased by $-0.04$ for GW200129, 0.12 for GW190701 and $-0.19$ for GW200208_22 (again for the aligned-spin quasi-circular case).

Finally, we have performed a computation to measure the probability that a GW event is eccentric. This method is more robust than simply considering odds ratios, as it combines information from the ensemble of GW events with astrophysical predictions. It is analogous to $p_{\mathrm{astro}}$ in LVK searches [131]. As more GWs are observed, this method will give tighter constraints on the rate of eccentric and quasi-circular events than simply using an astrophysical prior. This will lead to smaller uncertainty on the probability that individual events are eccentric. We find that $p_{\mathrm{ecc}} = 0.7490 - 0.9995$ for GW200129, $p_{\mathrm{ecc}} = 0.9763 - 0.9774$ for GW190701, and $p_{\mathrm{ecc}} = 0.7197 - 0.7286$ for GW200208_22 when using astrophysical priors. We extend this formalism to evaluate the probability that there exists an eccentric event in the population. We find this to be $p_{\mathrm{ecc,\,pop}} > 1 - 4.7 \times 10^{-3}$ when using `BayesWave A` glitch mitigation on GW200129 and $p_{\mathrm{ecc,\,pop}} > 1 - 8.4 \times 10^{-8}$ when using `gwsubtract` glitch mitigation on GW200129.

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### Appendix A: Eccentricity estimates from single detector networks

GW200129 was detected in LIGO Hanford, LIGO Livingston and VIRGO. In the main text, we perform inference using the combined data from LIGO Hanford and LIGO Livingston. However, due to the glitch in LIGO Livingston, it is useful to seperate the analysis into single detector analyses. This was also performed in Ref. [205]. As shown in Fig. 11, the majority of evidence for eccentricity comes from LIGO Livingston. There is still marginal support for eccentricity in the Hanford only analysis with $\log_{10}\mathcal{B}_{\rm EAS/QCAS} = 0.86$. Interestingly, the maximum likelihood of the Hanford only analysis peaks at $e_{10\rm Hz} = 0.40$.

The difference in the Bayes factors between Hanford and Livingston can be explained by the lack of SNR in Hanford. The SNR in Livingston is 22 whereas the SNR in Hanford is only 14. The difference in SNR is due to the particular antenna patterns for this event. We can take the max-likelihood waveform of GW200129 analyzed in both detectors and compute $F_+^2 + F_\times^2$ [142] to get an order of magnitude estimate of the impact of the detector projection. We find that for Hanford $F_+^2 + F_\times^2 = 0.53$ and for Livingston $F_+^2 + F_\times^2 = 0.92$.

We can then ask if the lack of evidence of eccentricity in Hanford is due to low SNR. To do this we perform an injection recovery using the max likelihood parameters of GW200129 analyzed with Hanford. The maximum likelihood point of the Hanford only analysis peaks at $e_{10\rm Hz} = 0.40$. If we decrease $d_L$ of the injection to artificially set the SNR to 22, we see a peak in the eccentricity posterior. However, performing the same injection with lower SNR yields a eccentricity posterior much closer to the prior. This implies the reason there is only a small bump in the eccentricity in Hanford is due to the lack of SNR in the detector rather than the lack of evidence for eccentricity.

### Appendix B: Lower Bound on $p_{\rm ecc,\,pop}$

In order to compute $p_{\rm ecc,\,pop}$ we must compute $p(\{g_i = 0\}\,|\,\{d_i\}, N)$. However, to do this we need to normalize the distribution $p(\{g_i\}\,|\,\{d_i\}, N)$. Note that

$$\begin{aligned} 1 &= \sum_{\{g_i\}} p(\{g_i\}|\{d_i\}, N) \\ &= \sum_{\{g_i\}} \int dR_{\rm E}\, dR_{\rm QC}\, p(\{g_i\}, R_{\rm E}, R_{\rm QC}|\{d_i\}, N). \end{aligned} \tag{B1}$$

We don't have the normalization over $p(\{g_i\}, R_{\rm E}, R_{\rm QC}|\{d_i\}, N)$, but we can re-write Eq. (B1) as

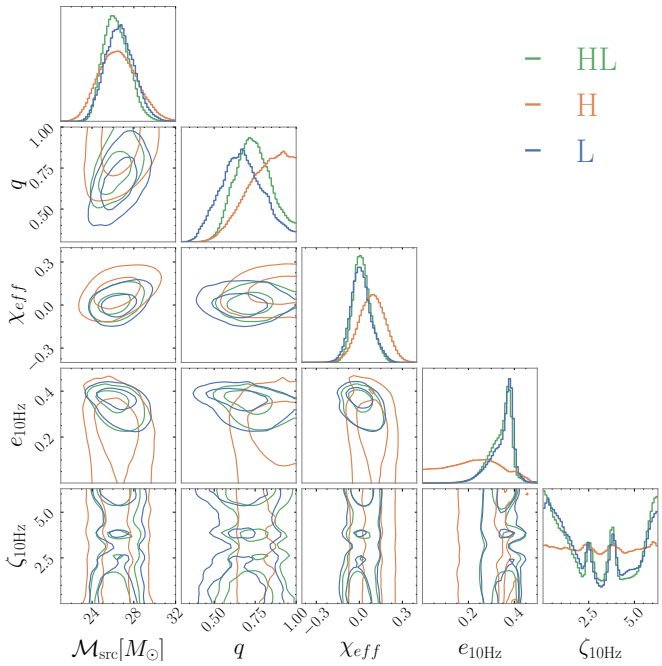


Figure 11. Analysis of GW200129 using LIGO-Hanford and LIGO-Livingston (HL), LIGO Hanford (H) and LIGO Livingston (L). We observe that the evidence for eccentricity comes largely from the LIGO-Livingston detector. The `gwsubtract` glitch subtraction is being used when using the Livingston detector. There is no glitch in the Hanford detector.

$$\beta = \sum_{\{g_i\}} \left[\prod_{\{i|g_i=1\}}^{N_{\rm E}} Z_{\rm E}(d_i)\right]\left[\prod_{\{i|g_i=0\}}^{N_{\rm QC}} Z_{\rm QC}(d_i)\right] I(\{g_i\}) \tag{B2}$$

where $\beta$ is the normalization of $p(\{g_i\}, R_{\rm E}, R_{\rm QC}|\{d_i\}, N)$ and $I(\{g_i\})$ is an evidence independent integral over the rates

$$I(\{g_i\}) = \int dR_{\rm E}\, dR_{\rm QC}\, R_{\rm E}^{\sum g_i} R_{\rm QC}^{\sum(1-g_i)} e^{-(\alpha_{\rm E} R_{\rm E} + \alpha_{\rm QC} R_{\rm QC})} p(R_{\rm E}, R_{\rm QC}), \tag{B3}$$

which depends only on the number of eccentric and quasi-circular flags and the prior over the rates.

In principle we can compute the sum in Eq. (B2) exactly. However, the sum has $2^N$ elements so if all the summands are important this will become infeasible. Instead, if we can identify dominant terms in the sum we can place a lower bound on $\beta$. Then we can we can put an upper bound on $p(\{g_i = 0\}\,|\,\{d_i\})$ which translates to a lower bound on $p_{\rm ecc,\,pop}$. So the larger the lower bound we can place on $\beta$ the larger the lower bound we can place on $p_{\rm ecc,\,pop}$. Thus our goal is to now eliminate small terms from the sum to get as large a lower bound as possible.

There are two scenarios to consider: using a uniform prior or using an astrophysical prior. If we use a uniform prior, each $I(\{g_i\})$ is of the same order in Eq. (B2). Thus the $\{g_i\}$

which contribute the most in the sum are those which select the events with $Z_{\rm E} \gg Z_{\rm QC}$ or $Z_{\rm E} \ll Z_{\rm QC}$. Given that we have no events with $Z_{\rm E} \ll Z_{\rm QC}$ (see Fig. 2) we focus on cases where $Z_{\rm E} \gg Z_{\rm QC}$. Let $\{j_1, \dots j_{N_{\rm E}}\}$ be the indexes of the $N_{\rm E}$ events with $Z_{\rm E} \gg Z_{\rm QC}$. We can remove any terms in Eq. (B2) which do not have at least one event with $Z_{\rm E} \gg Z_{\rm QC}$. That is we remove $\{g_i\}$ with

$$\sum_{i \in \{j_1, \dots j_{N_{\rm E}}\}} g_i = 0, \tag{B4}$$

from the sum. The normalization $\beta$ becomes

$$\begin{aligned}\beta > \sum_{k=1}^{N_{\rm E}} \sum_{\{j_k\}}^{\binom{N_{\rm E}}{k}} & \left[\prod_{j_l \in \{j_k\}} Z_{\rm E}(d_{j_l})\right] \\ & \sum_{\{g_i | i \notin \{j_k\}\}} \left[\prod_{\{i | g_i = 1 \wedge i \notin \{j_k\}\}} Z_{\rm E}(d_i)\right] \left[\prod_{\{i | g_i = 0 \wedge i \notin \{j_k\}\}} Z_{\rm QC}(d_i)\right] \\ & \times I(\{g_i\}).\end{aligned} \tag{B5}$$

Where it is implied that $I(\{g_i\})$ here explicitly sets the $j_l$ flags to 1. In the first sum we sum over the number of flags, $k$, which are set to 1 and have $Z_{\rm E} \gg Z_{\rm QC}$. The second sum represents the sum over the $N_{\rm E}$ choose $k$ combinations of flags with $Z_{\rm E} \gg Z_{\rm QC}$. Each, $\{j_k\}$ is shorthand for creating a set choosing $k$ elements from $\{j_1, \dots j_{N_{\rm E}}\}$. These first two sums serve to separate the events with $Z_{\rm E} \gg Z_{\rm QC}$ from the events with $Z_{\rm E} \sim Z_{\rm QC}$. Finally, the last sum is over the $\{g_i\}$ with $Z_{\rm E} \sim Z_{\rm QC}$.

This last sum is still intractable, but we can note that each term in the last sum of Eq. (B5) is at least greater than $I_s \prod_i Z_s(d_i)$ where $Z_s(d_i) = \min(Z_{\rm E}(d_i), Z_{\rm QC}(d_i))$ and $I_s = \min_{\{g_i\}} I(\{g_i\})$. $I_s$ can be computed directly from Eq. B3. We find that $I(\{g_i\})$ is minimized when $\sum g_i = N$ (in the uniform prior case this is also equivalent to the case where all the $\sum g_i = 0$ via symmetry). Thus we can place a bound on $\beta$ of

$$\begin{aligned}\beta > \sum_{k=1}^{N_{\rm E}} \sum_{\{j_k\}}^{\binom{N_{\rm E}}{k}} & \left[\prod_{j_l \in \{j_k\}} Z_{\rm E}(d_{j_l})\right] \\ & \times 2^{N - N_{\rm E}} \left[\prod_{i}^{N-k} Z_s(d_i)\right] I_s.\end{aligned} \tag{B6}$$

Here, the $2^{N-N_{\rm E}}$ comes from counting the number of terms in the last sum of Eq. B5. In our case with $N_{\rm E} = 3$, this is a sum over only $3 + 1 + 3 = 7$ elements.

However, if we use a astrophysical prior note $R_{\rm QC}$ has support for values much larger than $R_{\rm E}$ in the prior (see top left of Fig. 10). Thus since $I(\{g_i\})$ has terms like $R_{\rm QC}^{\sum 1 - g_i}$ in the integrand, the $\{g_i\}$ with the largest $\sum 1 - g_i$ will dominate Eq. (B2). Thus we can group terms which have all $g_i = 0$, terms which have all but one $g_i = 0$ terms which have all but two $g_i = 0$ and so on. In each grouping, the terms which contribute the most are those with at least one $g_{j_l} = 1$ with $j_l \in \{j_1 \dots j_k\}$. The lower bound then becomes

$$\beta > \sum_{k=0}^{N_{\rm E}} \sum_{\{j_k\}}^{\binom{N_{\rm E}}{k}} \left[\prod_{j_l \in \{j_k\}} Z_{\rm E}(d_{j_l})\right] \left[\prod_{i \notin \{j_k\}} Z_{\rm QC}(d_i)\right] I(\{g_i\}) \tag{B7}$$

## Appendix C: Kick velocity of GW200129

As discussed in the main text, when analyzing GW200129 with `SEOBNRv4EHM` we find a low kick velocity. We compute the kick velocity for `SEOBNRv4EHM` by integrating the momentum radiated throughout the orbit, $P^{\rm rad}$, using the waveform modes $h_{lm}$ [253]. We then determine the remnant mass, $m_f$, using a fit to numerical relativity as described in Ref. [129]. Then the remnant kick velocity is simply $v_f = P^{\rm rad}/m_f$. For `NRSur7dq4` we can simply use the `NRSur7dq4Remnant` model from Ref. [129] with the `surfinBH` package to find $v_f$.

However, obtaining a low kick velocity when analyzing GW200129 with `SEOBNRv4EHM` does *not* prove that this event has a negligible kick. The large kicks in BBHs come from asymmetric momentum emission. The highest momentum asymmetry occurs with misaligned spins [254, 255]. Thus with an aligned-spin model like `SEOBNRv4EHM` or `SEOBNRv4HM` the recoil velocity has to be small *by construction* [8]. But if one were to include both eccentricity and spin-precession, we may find that GW200129 still has a measurable kick velocity.

## Appendix D: Precessing injections with eccentric recovery and vice versa

Due to the possibility that eccentricity could be confused for spin-precession in short signals [123, 124], we perform an injection recovery campaign using `SEOBNRv4EHM` and `NRSur7dq4`. In this case we are most interested in GW200129. Therefore, we first analyze GW200129 with `gwsubtract` using both `SEOBNRv4EHM` and `NRSur7dq4`. We then find the maximum likelihood point from each analysis. Finally, we inject a precessing `NRSur7dq4` waveform with the maximum likelihood parameters and recover it with `SEOBNRv4EHM` (and vice versa). The results of this analysis are shown in Fig. 13. With the parameters considered here, we do not find evidence that spin-precession can be mistaken for eccentricity (or vice-versa).

[8] It has also been that in low eccentricity NR simulations kicks are only $\sim 25\%$ larger than the quasi-circular case [256].

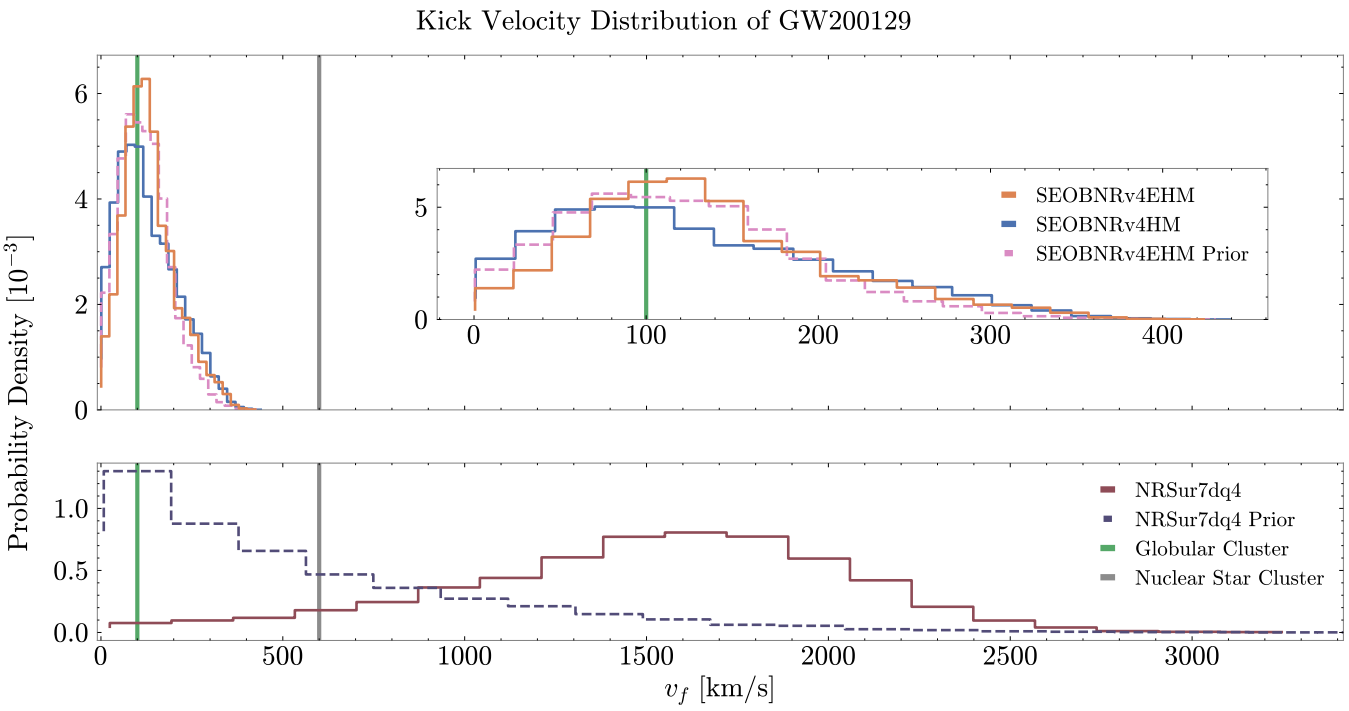


Figure 12. Kick velocity of GW200129 in km/s. Shown on the top (bottom) figure is the posterior distribution of the kick velocity using samples from the `SEOBNRv4EHM` and `SEOBNRv4HM` (`NRSur7dq4`) analyses using `gwsubtract` glitch mitigation. The green (grey) vertical lines show the typical escape velocity of a globular (nuclear) cluster as shown in Ref. [203].

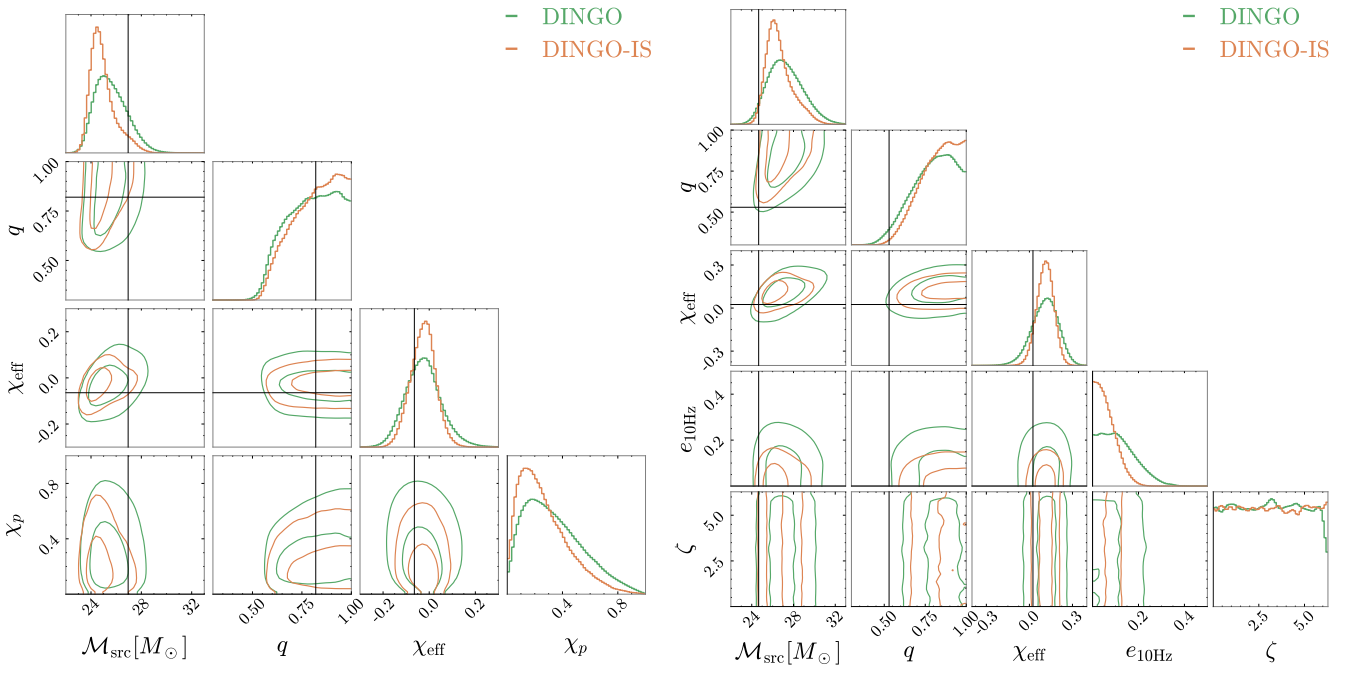


Figure 13. Posterior distributions obtained with `NRSur7dq4` (left) and `SEOBNRv4EHM` (right) when injecting `SEOBNRv4EHM` (left) or `NRSur7dq4` (right). The injection parameters were chosen to be the maximum likelihood point of GW200129 with `gwsubtract` glitch mitigation analyzed with `SEOBNRv4EHM` (left) and `NRSur7dq4` (right).

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