Part B : Gas Physics

1 Pressure and hydrostatic equilibrium

1.1 Pressure Forces

A defining characteristic of a gas is that the particles are collisional (unlike in a pure gravitational N-body system. An important manifestation of this collisionality is pressure; the pressure $\mathcal{P}$ of a gas is the force exerted per unit area.

We will look at the microscopic theory of gases later (such a theory is inevitably statistical in nature since we're dealing with the motions of a very large number of particles). Here we take this macroscopic definition of pressure and proceed to look at the role of pressure forces within a gaseous system and how we can establish equilibrium structures where gravity and pressure are in balance.

As always, the place to start is with the equation of motion. By examining a small patch of gas, we can show (IN CLASS) that the equation of motion for a gas is

$$\rho \frac{d\mathrm{v}}{dt} = -\nabla \mathcal{P} + \rho \mathbf{g}, \tag{1}$$

where ρ is the density of the gas and $\mathbf{g}$ is the gravitational field.

1.2 Hydrostatic Equlibrium

A particularly important situation arises when the pressure forces are in balance with gravity and hence there is no acceleration. If, furthermore, the gas starts off at rest (i.e. in a static configuration), the resulting equilibrium is called **hydrostatic equilibrium**.

Putting the acceleration (LHS) to zero and recalling $\mathbf{g} = -\nabla\Phi$ where Φ is the gravitational potential, we have the **equation of hydrostatic equilibrium**:

$$\nabla \mathcal{P} = -\rho \nabla \Phi. \tag{2}$$

This is a partial differential equation and, to solve it, we must have some relationship between $\mathcal{P}$and ρ. In detail, such a relationship is called the **equation of state** and depends upon the physics that is at play (i.e. whether the gas is adiabatic, isothermal, electron-degenerate, radiation-pressure dominated etc.) and requires a microscopic theory to calculate. We get to this later. For now, we just state some examples of equations of state:

- Isothermal (constant temperature) ideal gas, $\mathcal{P} = K\rho$ where $K = c_s^2$ (c_s^2 is isothermal sound speed),

- Adiabatic monatomic ideal gas, $\mathcal{P} = K\rho^{5/3}$,
- non-relativistic electron degeneracy pressure, $\mathcal{P} = K\rho^{5/3}$,
- Radiation-dominated gas, $\mathcal{P} = K\rho^{4/3}$

1.3 Atmospheres in externally imposed gravitational fields

A common situation is when you have a gas that is in hydrostatic equilibrium with a static gravitational field generated by some other mass. Then, $\nabla\mathcal{P} = -\rho\nabla\Phi$ where Φ is some specified function of position. This is described as an atmosphere in externally imposed gravitational field.

To illustrate some specific examples, let's adopt an isothermal equation of state. We can show that, in general, the solution to the equation of hydrostatic equilibrium is

$$\rho = \rho_0 e^{-\Phi/K}, \tag{3}$$

where ρ_0 is the value of the density where $\Phi = 0$.

IN CLASS, we examine two cases. One is a plane parallel isothermal atmosphere in a constant gravitational field (i.e. a good model for the Earth's atmosphere for heights $\ll$ the radius of the Earth. If z is the distance above the surface, the solution is

$$\rho = \rho_0 e^{-z/H}, \tag{4}$$

where we have defined a scale-height H that is given by

$$H = K/g. \tag{5}$$

The second case is considers the vertical structure of an accretion disk. We can show that the effective gravitational field is linear with height above the disk's mid plane, with a (quadratic) potential that takes the form

$$\Phi = f(R) + \frac{GMz^2}{2R^3}. \tag{6}$$

So the vertical density profile of an isothermal accretion disk is

$$\rho = \rho_0(R) e^{-z^2/H^2}, \tag{7}$$

where the scale-height is now given by

$$H^2 = \left(\frac{2R^3K}{GM}\right), \tag{8}$$

from which it can be shown that

$$\frac{H}{R} = \sqrt{2}\frac{c_s}{v_\phi} \tag{9}$$

1.4 Self-gravitating Atmospheres

Another important situation arises when the gas is in hydrostatic equilibrium with the gravitational field that it itself is generating. These are called self-gravitating atmospheres. To solve for the structure of these, we must simultaneously solve the equation of hydrostatic equilibrium and Poisson's equation (which describes the generation of the gravitational field):

$$\nabla\mathcal{P} = -\rho\nabla\Phi, \tag{10}$$

$$\nabla^2\Phi = 4\pi G\rho. \tag{11}$$

Again, let's specialize to an isothermal equation of state, $\mathcal{P} = c_s^2\rho$. We can combine these equations in order to derive a single second order PDE for the density,

$$c_s^2\nabla^2(\ln\rho) = 4\pi G\rho. \tag{12}$$

Let us assume that the system is spherically symmetric so that the density is purely a function of radius r (in fact, this will always be true provided that the boundary conditions are spherically symmetric!). Then eqn. 12 becomes,

$$\frac{c_s^2}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}(\ln\rho)\right] = 4\pi G\rho. \tag{13}$$

This is a second order ordinary differential equation for $\rho(r)$ and, to solve it, we in general need to specify two boundary conditions.

Let's examine one particularly simple solution. Let's hypothesize the existence of a solution which has no preferred scale and hence has a power-law form,

$$\rho(r) = \rho_0(r/r_0)^{-\alpha}. \tag{14}$$

Substituting this into eqn. 13 we find that such a solution exists provided that $\alpha = -2$ and that the solution must be

$$\rho(r) = \frac{c_s^2}{2\pi G r^2}. \tag{15}$$

This is known as the singular isothermal sphere since the solution diverges at the origin. In fact, there exists solutions to eqn. 13 which remain regular at the origin (they have flattened cores). Such solutions are called Bonnor-Ebert spheres.

2 Statistical Mechanics

2.1 Recap of the laws of thermodynamics

Before we get into our discussion of the detailed microscopic physics of gases, we briefly recap some thermodynamic principles. These are encapsulated in the four laws of thermodynamics.

0th Law of Thermodynamics : Heat diffuses from hot systems to cold systems. If system A and B are in thermal equilibrium with each other (no net heat flow despite being in thermal contact), then they are at the same temperature. If system A is in thermal equilibrium with both system B and C, then systems B and C are in thermal equilibrium with each other.

First Law of Thermodynamics : An equilibrium system is characterized by an internal energy U. When a system interacts with its surroundings, we have

$$dU = dQ + dW, \tag{16}$$

where dQ is the heat that flows into the system and $dW = -\mathcal{P}dV$ is the work done on the system.

Second Law of Thermodynamics : Any system can be characterized by its entropy S such that $dQ = TdS$. In any closed system, the entropy can only ever increase, $dS \geq 0$.

Third Law of Thermodynamics : The entropy of a system has a well defined limit S_0 (based exclusively on the configuration of its microscopic components) as one approaches absolute zero of temperature, $S \to S_0$ as $T \to 0$.

2.2 The Boltzmann distribution

One more fundamental preliminary that we ned to touch upon, which shall be stated without proof. Suppose that a system is in thermal equilibrium with its surroundings. Furthermore, suppose that the system is allowed to exist in one of a number of states with energies E_1, E_2, E_3... Then the probability of the system being found in various energy states satisfies the Boltzmann distribution,

$$\frac{p(E_1)}{p(E_2)} = \frac{e^{-E_1/kT}}{e^{-E_2/kT}}. \tag{17}$$

2.3 The statistical mechanics of an ideal gas

Finally, let's look at the microscopic behavior of an ideal gas. We will assume that the gas is composed of monatomic ("structureless") particles with mass m. The probability of finding a particular particle in a state with velocity components $v_x \to v_x + dv_x$, $v_y \to v_y + dv_y$, $v_z \to v_z + dv_z$ is

$$dp = f(\mathbf{v})\, dv_x\, dv_y\, dv_z, \tag{18}$$

where we have defined the distribution function $f(\mathbf{v})$ that takes on the form of a Boltzmann distribution,

$$f(\mathbf{v}) = \frac{1}{(2\pi kT/m)^{3/2}} e^{-\frac{1}{2}m(v_x^2+v_y^2+v_z^2)/kT}. \tag{19}$$

If there are a total of n particles per unit volume, then we can show the following important results:

$$u = \int\int\int \frac{1}{2}mv^2\, nf(\mathbf{v})\, dv_x\, dv_y\, dv_z = \frac{3}{2}nkT \tag{20}$$

$$\mathcal{P} = \int\int\int mv_x^2\, nf(\mathbf{v})\, dv_x\, dv_y\, dv_z = nkT, \tag{21}$$

where u is the internal energy density, $u = U/V$.

We can now see where some of the equations of state that we discussed in Section 1 come from. If the gas is isothermal (constant T) then,

$$\mathcal{P} = nkT = \frac{kT}{m}nm = \frac{kT}{m}\rho. \tag{22}$$

Thus, $\mathcal{P} \propto \rho$. If, instead, the gas is adiabatic, we have that $dS = 0$, i.e. there is no heat flowing into or out of the system. The First Law of Thermodynamics then reads,

$$dU = -\mathcal{P}dV. \tag{23}$$

But from the derivation above, we know that

$$\mathcal{P} = \frac{2}{3}u = \frac{2U}{3V}. \tag{24}$$

If you substitute this into the first-law, and then integrate, you can show that

$$\mathcal{P} = K\rho^{5/3}. \tag{25}$$

What is the constant of proportionality, K? Using the First Law of Thermodynamics, it can be shown that the entropy of an ideal gas is

$$S = Nk\ln(\mathcal{P}\rho^{-5/3}) + S_0, \tag{26}$$

and so the constant of proportionality is related to the entropy of the gas,

$$K \equiv \frac{\mathcal{P}}{\rho^{5/3}} = e^{(S-S_0)/Nk}. \tag{27}$$

2.4 γ-law gases

The monatomic ideal gas has $u = \frac{3}{2}\mathcal{P}$. In a useful generalization, consider a gas that has

$$u = \frac{1}{\gamma - 1}\mathcal{P}. \tag{28}$$

A gas that obeys this condition is known as a γ-law gas. Using the exact same techniques as above, we can show that adiabatic changes of a γ-law gas have $\mathcal{P} \propto \rho^{\gamma}$.

2.5 Radiation gases

Let's now consider another kind of gas — radiation, i.e., a gas of photons. This differs from the ideal gas discussed previously in two important ways:

1. Photons always have $|\mathbf{v}| = c$. Thus, we shall label photons with their momenta (p_x, p_y, p_z) rather than their velocities. Recall that the energy and momentum of is related through $E = |\mathbf{p}|c$.

2. The total number of photons can change — photons can be created or destroyed.

So, we shall shift notation a little and say that the number density of photons in the momentum range $p_x \to p_x + dp_x, p_y \to p_y + dp_y, p_z \to p_z + dp_z$ is

$$f_R(\mathbf{p})\, dp_x\, dp_y\, dp_z. \tag{29}$$

Just as before, the distribution function $f_R(\mathbf{p})$ is given by a Boltzmann factor, but it is modified by quantum effects. However, we can deduce a lot about the properties of "thermal radiation" (i.e. a radiation field that is in thermal equilibrium with its surroundings) without knowing the exact functional form of $f_R(\mathbf{p})$. Thermal radiation is isotropic (i.e. the distribution of momenta is isotropic) so that

$$f_R(\mathbf{p}) = f_R(p), \qquad \text{where } p \equiv \mathbf{p}. \tag{30}$$

With this knowledge, we can show that the internal energy density of the radiation is

$$u_R = \int\int\int E f_R(\mathbf{p})\, dp_x\, dp_y\, dp_z = 4\pi c \int_0^\infty p^3 f_R(p)\, dp. \tag{31}$$

We can also show that the pressure is given by

$$\mathcal{P}_R = \int\int\int c p_x \cos\theta f_R(\mathbf{p})\, dp_x\, dp_y\, dp_z = \frac{4\pi c}{3} \int_0^\infty p^3 f_R(p)\, dp. \tag{32}$$

Thus, we can see that

$$\mathcal{P}_R = \frac{1}{3} u_R. \tag{33}$$

This is consistent with our "γ-law" gas form with $\gamma = 4/3$.

We've already stated that photons can be created and destroyed. Thus, when a radiation field is in thermal equilibrium, its properties such as the energy density can be a function of temperature only, i.e., $u_R = u_R(T)$. From the First Law of Thermodynamics, we use this fact to show that

$$u = aT^4, \tag{34}$$

where a is the *radiation constant* and has the value $a = 7.566 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$.

2.6 An application to cosmology

We now apply some of these ideas to cosmology. Today, the total energy density of the Universe (neglecting Dark Energy) is

$$\epsilon = \rho c^2 + \frac{3}{2}nkT + aT^4 \tag{35}$$

Let's examine the magnitude of each term.

1. Rest-mass energy, $\epsilon_{\rm rest} = \rho c^2 = (2.5 \times 10^{-27}\,{\rm kg\,m^{-3}})c^2 \approx 2.2 \times 10^{-10}\,{\rm J\,m^{-3}}$,
2. Thermal energy, $\epsilon_{\rm thermal} = \frac{3}{2}nkT = \frac{3}{2}(1.5\,{\rm m^{-3}})k(10^5\,{\rm K}) \approx 3.1 \times 10^{-18}\,{\rm J\,m^{-3}}$,
3. CMB energy, $\epsilon_{\rm rad} = aT^4 = a(2.7\,{\rm K})^4 = 4.0 \times 10^{-14}\,{\rm J\,m^{-3}}$.

So, today, the rest mass energy of the matter dominates the energy content of the Universe. But, was this always true?

Consider a "co-moving" volume of the Universe which is a large cube of side-length a. Then, we can show that

$$\epsilon_{\rm rest} \propto a^{-3}, \tag{36}$$

$$\epsilon_{\rm rad} \propto a^{-4}, \tag{37}$$

which means that $\epsilon_{\rm rad}/\epsilon_{\rm rest} \propto a^{-1}$. Hence, radiation becomes more and more important as we go back in time and the Universe was smaller.

3 Hydrodynamics

We have talked about static gas distributions (hydrostatic equilibrium) and as well as evolving uniform gas distributions. We now briefly talk about gas flows that have both structure and evolution — this is the subject of hydrodynamics.

3.1 The equations of hydrodynamics

We have already come across the equation of motion for the fluid:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla \mathcal{P} + \rho \mathbf{g}. \tag{38}$$

The time-derivative on the left-hand side is actually a bit more complex than it looks since we're talking about a vector field within the fluid and the acceleration can have a spatial as well as time dependence. Application of the chain rule shows that

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}, \tag{39}$$

giving us what is known as the Euler equation:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla \mathcal{P} + \rho \mathbf{g}. \tag{40}$$

We also know that he fluid must obey conservation of mass. By considering the flux of mass into a volume V, we can show that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{41}$$

known as the continuity equation.

Together with an equation of state and some prescription for the gravitational field $\mathbf{g}$, these are known as the equations of hydrodynamics.

3.2 Sound waves

Let's see how we can use these equations to deduce something interesting — the existence of sound waves. Consider a uniform medium at rest ($\mathbf{v} = 0$) with density ρ_0 and neglect gravity ($\mathbf{g} = 0$). Now consider some small disturbance such that

$$\rho = \rho_0 + \rho_1, \tag{42}$$

$$\mathbf{v} = \mathbf{v}_1 \tag{43}$$

$$\tag{44}$$

with ρ_1 and $\mathbf{v}_1$ "small". We also assume an isothermal equation of state throughout : $\mathcal{P} = K\rho$. If we substitute this into the equations of hydrodynamics and keep only the first order terms in the small quantities, we get

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -K\nabla\rho_1, \tag{45}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot (\mathbf{v}_1). \tag{46}$$

These can be combined to give

$$\frac{\partial^2 \rho_1}{\partial t^2} - K\nabla^2\rho_1 = 0 \tag{47}$$

Equation 47 is known as the wave equation. To see why, consider the case with one spatial dimension:

$$\frac{\partial^2 \rho_1}{\partial t^2} - K\frac{\partial^2 \rho_1}{\partial x^2} = 0 \tag{48}$$

If we define new variables,

$$\xi = x - K^{1/2}t, \tag{49}$$

$$\eta = x + K^{1/2}t, \tag{50}$$

$$\tag{51}$$

we can show that the 1-d wave equation becomes

$$\frac{\partial^2 \rho}{\partial \xi \, \partial \eta} = 0, \tag{52}$$

which means that the solution has the form

$$\rho_1 = F(\xi) + G(\eta) \tag{53}$$

where F and G can be any arbitrary functions. So, this describes the "translation" of the functions F and G along the curves $x = \pm K^{1/2} t$, i.e. a wave-like disturbance propagating at speed $c_s \equiv K^{1/2}$.