5. Three body and 2+1 body systems

5.1 General comments

For systems with more than two bodies, we can write the general equation of motion:

$$\ddot{\underline{r}}_i = -G \sum_{j \neq i} m_j \frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|^3}$$

However, for $n \geq 3$ bodies, there is no analytic solution to this set of equations! Solutions have to be obtained computationally.

<u>CLASS QUESTION</u>: How would you go about writing a computer code that could solve an N-body system?

In general, for $n \geq 3$ bodies the system is subject to <u>chaos</u>, i.e. a tiny perturbation of the position or velocities of a particle can induce exponentially diverging orbital trajectories. Chaos can be particularly prevalent in 3-body systems.

But, even in three body systems, we can find some special cases where the behavior can be non-chaotic. Important example: hierarchical triples:

QUESTION: What is behavior of this system?

Another important case is when one body has a much smaller mass than the other two, eg

- planet in a binary star system
- asteroid in the Sun-Jupiter system.

This is called the 2+1 (or reduced three body) problem.

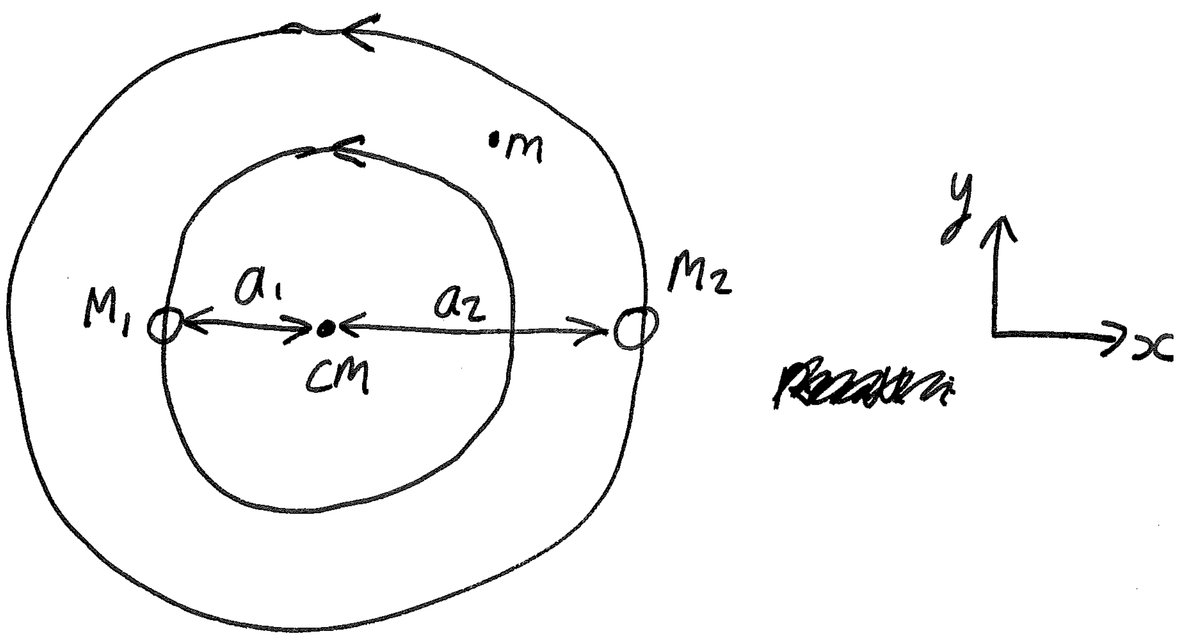
5.2: 2+1 body problem

Suppose that there are two massive bodies (M_1 and M_2) and a body with negligible mass (m). We can take m to be "test particle" ~~acting~~ moving within the gravitational fields of M_1 and M_2. This means that we ignore the gravitational force of m on the motions of M_1 and M_2.

The simplest and most commonly applicable case is when M_1 and M_2 are in circular orbits. QUESTION: WHY?

Lets analyze this system. It is easiest to work in the rotating reference frame with the center of mass at the origin

M_1 and M_2 are fixed in this coordinate system.



Let $a = a_1 + a_2$

Recall from last class : $\Omega_0 = \left(\frac{G(M_1+M_2)}{a^3}\right)^{1/2}$.

Now, gravitational force felt by mass m is

$$\underline{F}_{grav} = -\frac{GM_1 m}{|\underline{r}-\underline{r}_1|^3}(\underline{r}-\underline{r}_1) - \frac{GM_2 m}{|\underline{r}-\underline{r}_2|^3}(\underline{r}-\underline{r}_2)$$

The equation of motion for mass m in this rotating reference frame is

$$\ddot{\underline{r}} = \underbrace{-2\Omega\hat{\underline{z}}\times\dot{\underline{r}}}_{\text{Coriolis force}} \underbrace{-\Omega^2\hat{\underline{z}}\times(\hat{\underline{z}}\times\underline{r})}_{\text{Centrifugal force}} + \underbrace{\underline{F}_{grav}/m}_{\text{"real" force}}$$

Lets write this in terms of our (rotating) Cartesian coordinate system... note that

$$-2\Omega\hat{\underline{z}}\times\dot{\underline{r}} = -2\Omega\dot{x}\hat{\underline{y}} + 2\Omega\dot{y}\hat{\underline{x}}$$
$$-\Omega^2\hat{\underline{z}}\times(\hat{\underline{z}}\times\underline{r}) = \Omega^2\underline{r} = \Omega^2 x\hat{\underline{x}} + \Omega^2 y\hat{\underline{y}}$$

So,

$$\ddot{x} = 2\Omega\dot{y} + x\Omega^2 - \frac{GM_1(x+a_1)}{[(x+a_1)^2+y^2]^{3/2}} - \frac{GM_2(x-a_2)}{[(x-a_2)^2+y^2]^{3/2}} \quad ①$$

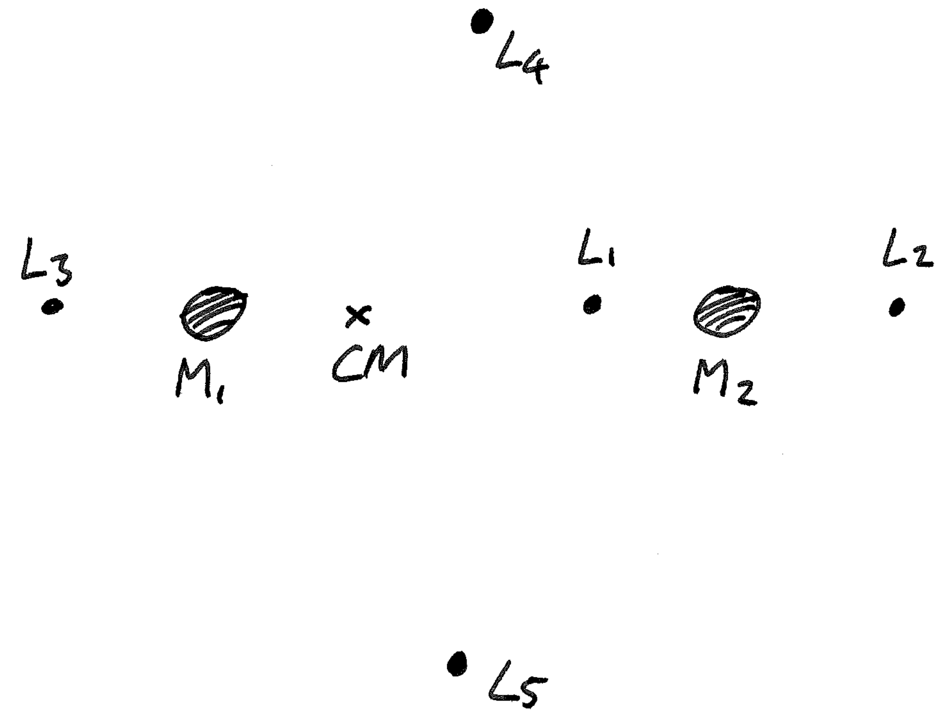
$$\ddot{y} = -2\Omega\dot{x} + y\Omega^2 - \frac{GM_1 y}{[(x+a_1)^2+y^2]^{3/2}} - \frac{GM_2 y}{[(x-a_2)^2+y^2]^{3/2}} \quad ②$$

In general, m gets buffeted around by gravitational fields of M_1 & M_2 and these equations tell us exactly how that buffeting occurs. But – are there any special places where there is no motion of m, ie, all terms cancel.

We seek places where, if $\dot{x} = \dot{y} = 0$ then $\ddot{x} = \ddot{y} = 0$.

~~If $x \neq 0$ and $y = 0$ then ② is automatically satisfied. Eqn ①~~

If we set $\dot{x} = \dot{y} = 0$ and $\ddot{x} = \ddot{y} = 0$, we find five points where eqns ① + ② are satisfied :-



These are known as the <u>Lagrange points</u>.

5.3 : Properties of the Lagrange points

Eqns of motion ① + ② become

$$x\Omega^2 - \frac{GM_1(x+a_1)}{[(x+a_1)^2+y^2]^{3/2}} - \frac{GM_2(x-a_2)}{[(x-a_2)^2+y^2]^{3/2}} = 0 \quad ③$$

$$y\Omega^2 - \frac{GM_1 y}{[(x+a_1)^2+y^2]^{3/2}} - \frac{GM_2 y}{[(x-a_2)^2+y^2]^{3/2}} = 0 \quad ④$$

Consider a point halfway in x between M_1 + M_2.

$$x = a_2 - a/2$$

$$x + a_1 = x + a - a_2 = a/2$$

$$x - a_2 = -a/2$$

Substituting into ④ gives

$$\Omega^2 - \frac{GM_1}{[a^2/4+y^2]^{3/2}} - \frac{GM_2}{[a^2/4+y^2]^{3/2}} = 0$$

$$\Rightarrow \frac{G(M_1+M_2)}{a^3} - \frac{G(M_1+M_2)}{[a^2/4+y^2]^{3/2}} = 0 \qquad \text{since } \Omega^2 = \frac{G(M_1+M_2)}{a^3}$$

$$\Rightarrow y = \frac{\sqrt{3}}{2}a$$

Feed this back into ③ ...

$$LHS = (a_2 - a/2)\frac{G(M_1+M_2)}{a^3} - \frac{GM_1}{a^3}\,a/2 + \frac{GM_2\,a/2}{a^3}.$$

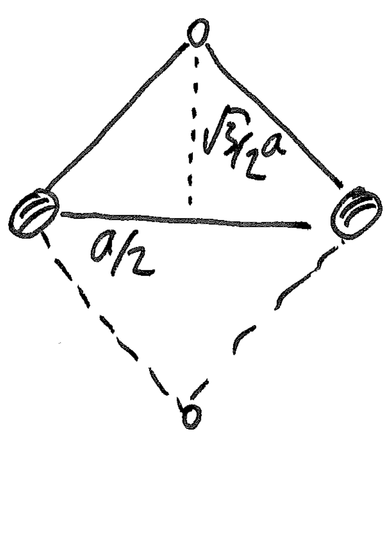
$$= (a_2 - \tfrac{a}{2})\frac{G(M_1+M_2)}{a^3} - \frac{G(M_1-M_2)}{a^3}\frac{a}{2}$$

But $a_2 = \dfrac{\mu}{M_2}a = \dfrac{M_1}{(M_1+M_2)}a$

$$\therefore LHS = \frac{G}{a^3}\left[aM_1 - \frac{a}{2}(M_1+M_2) - \frac{a}{2}(M_1-M_2)\right]$$

$$= 0$$

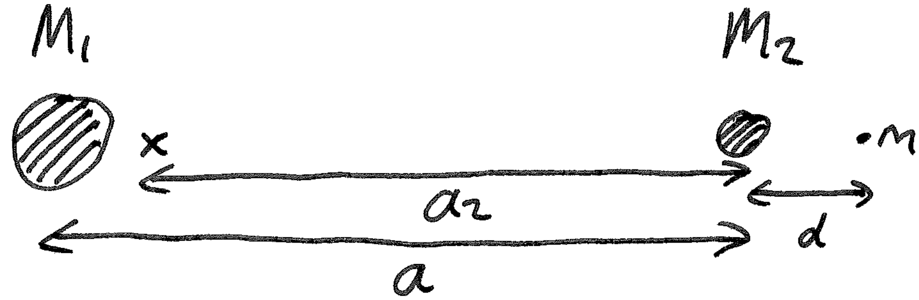
∴ Eqn ③ is satisfied. Hence the "out of line" Lagrange points are located at $(x, y) = (a_2 - a/2, \pm\sqrt{3}a/2)$



Thus, L_4 and L_5 make equilateral triangles with masses M_1 & M_2.

The other three Lagrange points are all in line ($y=0$) but, in general, there are no simple "closed form" expressions for them. Particular important case, however, is when $M_1 \gg M_2$ and in that case we can find approximations for the locations of the Lagrange points, esp L_1 & L_2.

So, lets consider case $M_1 \gg M_2$:-



Let $x = a_2 + d$, $|d|/a \ll 1$

Recall
$$a_2 = \frac{\mu}{M_2} a = \frac{M_1}{M_1+M_2} a$$

Posn relative to M_1 is $x + a_1 = d + a$
" " " M_2 " $x - a_2 = d$

At Lagrange points, we know that eqn ③ is satisfied :-

$$\frac{G(M_1+M_2)}{a^3} x = \frac{GM_1(x+a_1)}{[(x+a_1)^2]^{3/2}} + \frac{GM_2(x-a_2)}{[(x-a_2)^2]^{3/2}}$$

$$\Rightarrow \frac{G(M_1+M_2)}{a^3}\left[\frac{M_1}{M_1+M_2} a + d\right] = \frac{GM_1}{(a+d)^2} \pm \frac{GM_2}{d^2}$$

since $\frac{d}{|d|} = \pm 1$.

$$\Rightarrow \frac{1}{a^2}\left[M_1 + (M_1+M_2)\frac{d}{a}\right] = \frac{M_1}{a^2(1+d/a)^2} \pm \frac{M_2}{d^2}$$

$$\Rightarrow \frac{M_1}{a^2}\left[1 + \left(1+\frac{M_2}{M_1}\right)\frac{d}{a}\right] = \frac{M_1}{a^2(1+d/a)^2} \pm \frac{M_2}{d^2}$$

$$\Rightarrow \cancel{M_1}\ \frac{d^2}{a^2}\left[1+\left(1+\frac{M_2}{M_1}\right)\frac{d}{a}\right] = \frac{d^2}{a^2}\,\frac{1}{(1+d/a)^2} \pm \frac{M_2}{M_1}.$$

$$\Rightarrow \frac{d^2}{a^2} + \left(1+\frac{M_2}{M_1}\right)\frac{d^3}{a^3} \doteq \frac{d^2}{a^2}\underbrace{\left(1-\frac{2d}{a}\right)}_{\text{Taylor expansion}} \pm \frac{M_2}{M_1}$$

$$\Rightarrow \frac{d^3}{a^3} + \underbrace{\frac{M_2}{M_1}\frac{d^3}{a^3}}_{\text{small}} = -\frac{2d^3}{a^3} \pm \frac{M_2}{M_1}$$

$$\Rightarrow 3d^3/a^3 = \pm\frac{M_2}{M_1}$$

$$\Rightarrow d = \pm a\left(M_2/3M_1\right)^{1/3}$$

This is known as the Roche radius or the radius of the Hills sphere and, in some sense, delineates the sphere of influence of the small body M_2.

What about the dynamics of test particles close to the Lagrange points. This can be addressed using perturbation analysis (similar to epicyclic motion problem). We find that L_1, L_2 and L_3 are all unstable. L_4 and L_5 are stable provided that $M_2 \lesssim M_1/26$.

5.4 Effective Potential

Step back to the original equation of motion...

$$\underline{\ddot{r}} = -2\Omega\,\underline{\hat{z}}\times\underline{\dot{r}} + \Omega^2\underline{r} + \underline{F}_{grav}/m.$$

In terms of the gravitational potential, we know that

$$\underline{F}_{grav} = -m\nabla\Phi$$

Also, a mathematical identity is

$$\nabla\left(\tfrac{1}{2}\Omega^2 r^2\right) = \Omega^2\underline{r}$$

So, we can write

$$\underline{\ddot{r}} = -2\Omega\,\underline{\hat{z}}\times\underline{\dot{r}} + \nabla\left(\tfrac{1}{2}\Omega^2 r^2\right) \overset{-}{\cancel{+}} \nabla\Phi$$

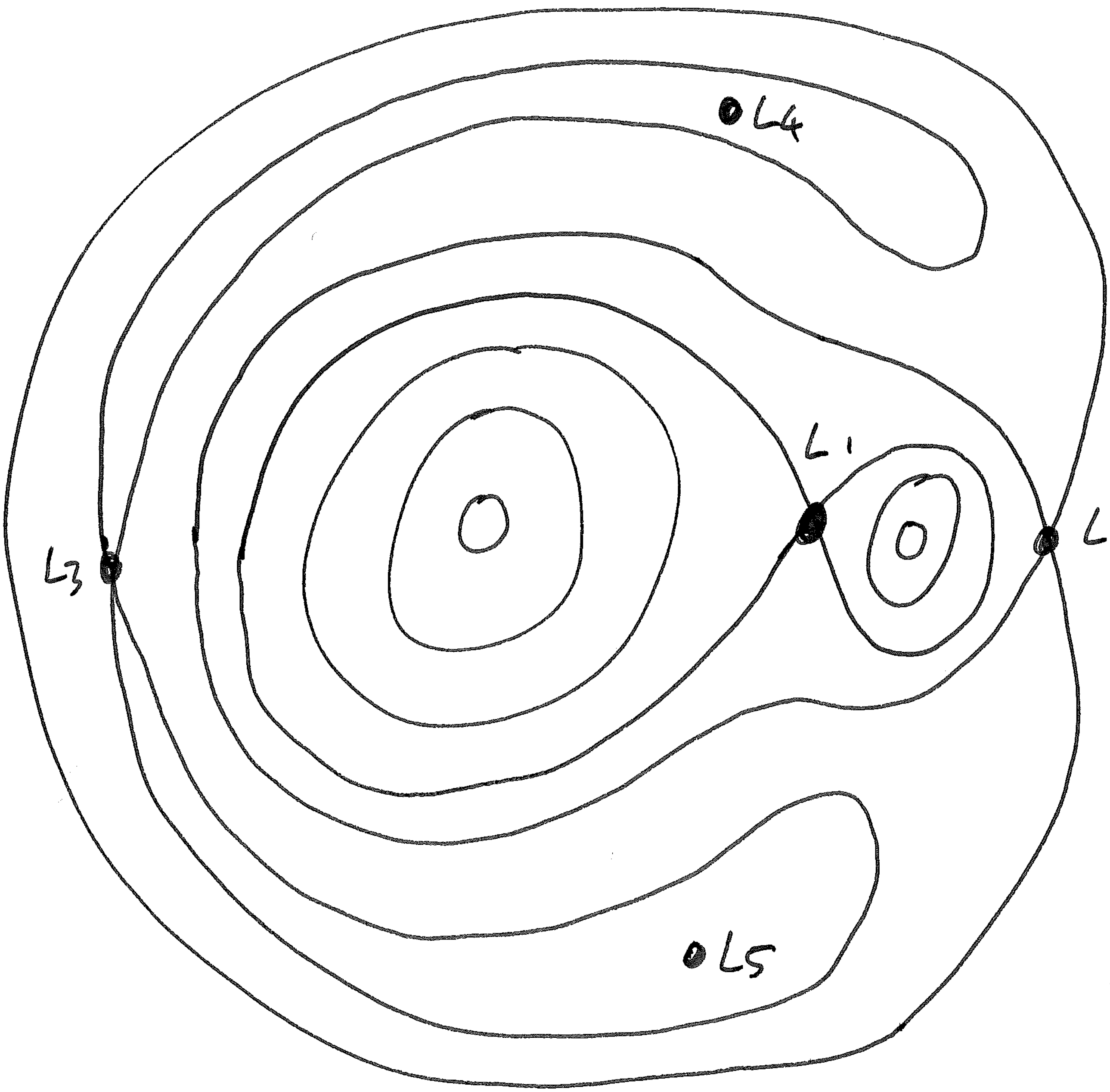
$$\Rightarrow\ \underline{\ddot{r}} = -2\Omega\,\underline{\hat{z}}\times\underline{\dot{r}} + \nabla\Phi_{eff}$$

where $\Phi_{eff} = \Phi - \frac{1}{2}\Omega^2 r^2$ is the *effective* potential

$$\Phi_{eff} = -\frac{GM_1}{|\underline{x}-\underline{x}_1|} - \frac{GM_2}{|\underline{x}-\underline{x}_2|} - \frac{\Omega_0^2}{2}(x^2+y^2)$$

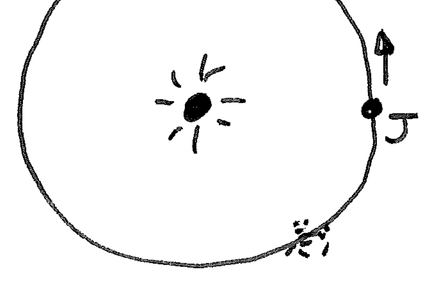
The gradient of the effective potential gives the acceleration of the test mass if it is at rest (and hence Coriolis term is zero). In terms of eff. potential, the Lagrange points are given by the condition $\nabla\Phi_{eff} = 0$.

However, stability is more subtle. For $M_2 \ll M_1/26$, L_4 and L_5 are stable *despite* being maxima of Φ_{eff}.

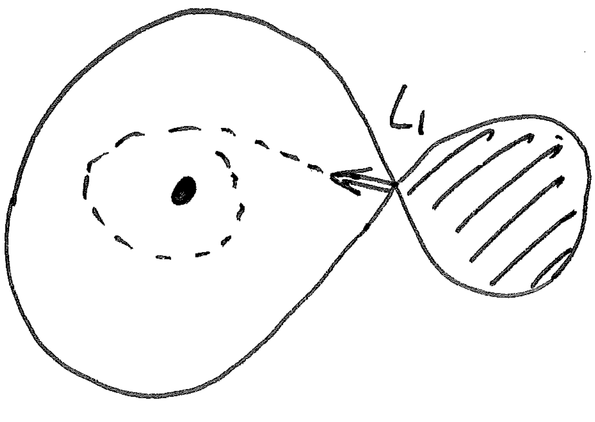


5.5 Astrophysical significance of Lagrange points

(a) Trojan asteroids - a group of asteroids "trapped" at L_4 and L_5 points of Sun-Jupiter system.



(b) For close binary star systems, the L_1 point is significant as the position where mass can flow from one star onto another. Eg, black hole system



Donor star fills the critical equipotential surface.

Matter streams through L_1 point and is then deflected by Coriolis forces.

Also, CV - mass transfer onto WD

Algol binaries - mass transfer onto main sequence star

(c) L_2 of the Sun-Earth system is important for space-science. It is an excellent place to "park" a deep space satellite. Used by several current and future missions.

L_2 is unstable and so some gentle nudging needed to keep it in orbit.