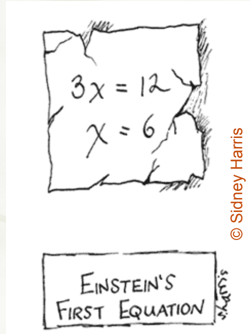
Lecture 16 : Cosmological Models II

- Deceleration parameter
- Beyond standard cosmological models
  - The Friedman equation with Λ
  - Effects of nonzero Λ
- Solutions for special cases
  - de Sitter solution
  - Static model
  - Steady state model



Recap

- The **Friedmann equation** is obtained by plugging each of the possible FRW metric cases into Einstein's GR equation
- Result is an equation saying how the cosmic scale factor *R(t)* must change in time:

$$\left(\frac{dR}{dt}\right)^2 = H^2R^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- The Friedmann equation can also be written as:

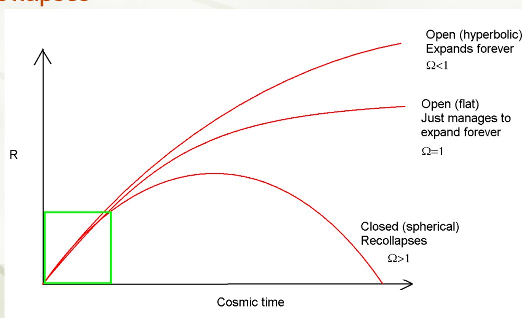
$$\Omega = 1 + \frac{kc^2}{H^2R^2}$$

- where

$$\Omega \equiv \frac{\rho}{\rho_{crit}} \equiv \frac{\rho}{(3H^2/8\pi G)}$$

Recap

- Whether case with *k=-1,0, or 1* applies depends on the ratio of the actual density to the “critical” density, Ω
- Properties of standard model solutions:
  - k=-1, Ω<1 expands forever
  - k=0, Ω=1 “just barely” expands forever
  - k=+1, Ω>1 expands to a maximum radius and then recollapses



The deceleration parameter, *q*

- The **deceleration parameter** measures how quickly the universe is decelerating (or accelerating), i.e. how much *R(t)* graph curves
- In standard models, deceleration occurs because the gravity of matter slows the rate of expansion
- For those comfortable with calculus, actual definition of *q* is:

$$q = -\frac{1}{RH^2}\frac{d^2R}{dt^2}$$

- In the textbook, d^2R/dt^2 is written as $\ddot{R}$, pronounced “R double-dot”

Matter-only standard model

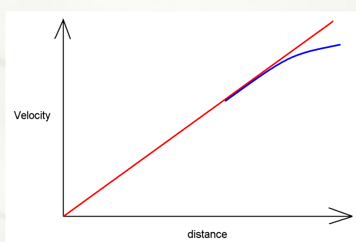
- In standard model where density ρ is entirely from the rest mass energy of matter, it turns out that the value of the deceleration parameter is given by

$$q = \frac{\Omega}{2}$$

- This gives a consistency check for the standard, matter-dominated models… we can attempt to measure Ω in two ways:
  - Direct measurement of how much mass is in the Universe --i.e. measure mass density ρ , measure Hubble parameter *H*, and then compare ρ to the critical value $\rho_{crit}=3H^2/(8\pi G)$
  - Use measurement of deceleration parameter, *q*
  - Measurement of *q* is analogous to measurement of Hubble parameter, by observing change in expansion rate as a function of time: need to look at how *H* changes with redshift for distant galaxies

Direct observation of *q*

- Deceleration shows up as a deviation from Hubble's law…



- A very subtle effect – have to detect deviations from Hubble's law for objects with a large redshift

Obtaining Friedmann eq. for mass-only Universe from Newtonian theory

- Consider spherical piece of the Universe large enough to contain many galaxies, but much smaller than the Hubble radius (distance light travels in a Hubble time).
  - Radius is r. Mass is $M(r)=4\pi r^3\rho/3$
- Consider particle m at edge of this sphere; it feels gravitational force from interior of sphere, $F=-GM(r)m/r^2$
- Suppose outer edge, including m, is expanding at a speed

  $v(r)= \Delta r/\Delta t=dr/dt$
- Then, from Newton's 2nd law, rate of change of v is the acceleration $a =\Delta v/\Delta t=dv/dt$, *with* $F=ma$, yielding

  $a= F/m =-GM(r)/r^2 =-4\pi Gr\rho/3$
- Using calculus, this can be worked on to obtain

  $v^2 =8\pi G\rho r^2/3 + constant$

- But if we simply identify the "constant" (which is twice the Newtonian energy) with $-kc^2$, and reinterpret r as R, the cosmic scale factor, we have Friedmann's equation!

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- Newtonian interpretation is therefore:
  - k=-1 is "positive energy" universe (which is why it expands forever)
  - k=+1 is "negative energy" universe (which is why it recollapses at finite time)
  - k=0 is "zero energy" universe (which is why it expands forever but slowly grinds to a halt at infinite time)
- Analogies in terms of throwing a ball in the air…

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Expansion rates

- For flat (k=0, Ω=1), matter-dominated universe, it turns out there is a simple solution to how *R* varies with *t* :

$$R(t) = R(t_0)\left(\frac{t}{t_0}\right)^{2/3}$$

  - This is known as the Einstein-de Sitter solution
  - For this solution, $V=\Delta R/\Delta t=(2/3)(R(t_0)/t_0)(t/t_0)^{-1/3}$
    - How does this behave for large time? What is $H=V/R$?
- In solutions with Ω>1, expansion is slower (followed by recollapse)
- In solutions with Ω<1, expansion is faster

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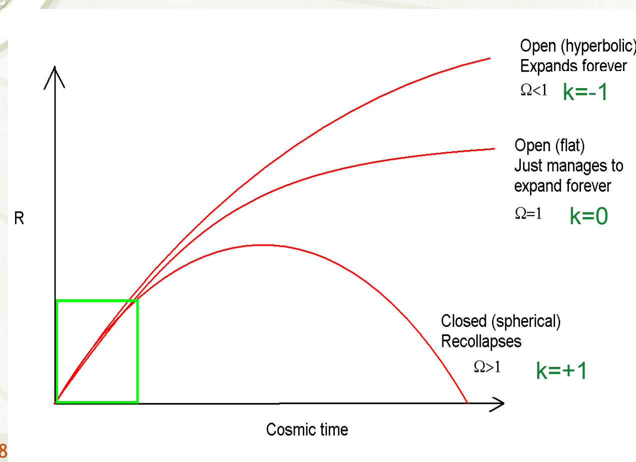
- Open, flat, and closed solutions result for different values of Ω



Modified Einstein's equation

- But Einstein's equations most generally also can include an extra constant term; i.e. in

$$\underline{\underline{G}} = \frac{8\pi G}{c^4}\underline{\underline{T}}$$

the **T** term has an additional term which just depends on space-time geometry times a constant factor, Λ

- This constant Λ (Greek letter "Lambda") is known as the "***cosmological constant***";
- Λ corresponds to a "vacuum energy", i.e. an energy **not** associated with either matter or radiation
- Λ could be positive or negative
  - Positive Λ would act as a repulsive force which tends to make Universe expand faster
  - Negative Λ would act as an attractive force which tends to make Universe expand slower
- Energy terms in cosmology arising from positive Λ are now often referred to as "***dark energy***"

Modified Friedmann Equation

- When Einstein equation is modified to include Λ, the Friedmann equation governing evolution of $R(t)$ changes to become:

$$\left(\frac{dR}{dt}\right)^2 = H^2R^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3} - kc^2$$

- Dividing by $(HR)^2$, we can consider the relative contributions of the various terms evaluated at the present time, t_0
- The term from matter at t_0 has subscript "M";
- Two additional "Ω" density parameter terms at t_0 are defined:

$$\Omega_M \equiv \frac{\rho_0}{\rho_{crit}} \equiv \frac{\rho_0}{(3H_0^2/8\pi G)} \qquad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2} \qquad \Omega_k \equiv -\frac{kc^2}{R_0^2H_0^2}$$

- Altogether, at the present time, t_0, we have

$$1 = \Omega_M + \Omega_\Lambda + \Omega_k$$

Generalized Friedmann Equation in terms of Ω's

- The generalized Friedmann equation governing evolution of $R(t)$ is written in terms of the present Ω's (density parameter terms) as:

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0^2R_0^2\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \Omega_k\right]$$

- The only terms in this equation that vary with time are the scale factor R and its rate of change dR/dt
- Once the constants H_0, Ω_M, Ω_Λ, Ω_k are measured empirically (using observations), then whole future of the Universe is determined by solving this equation!
- Solutions, however, are more complicated than when $\Lambda=0$ …

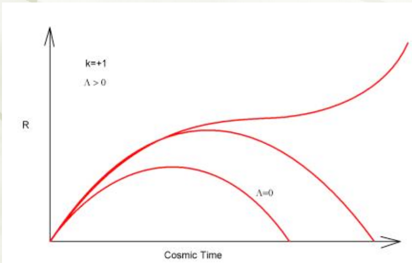
Effects of Λ

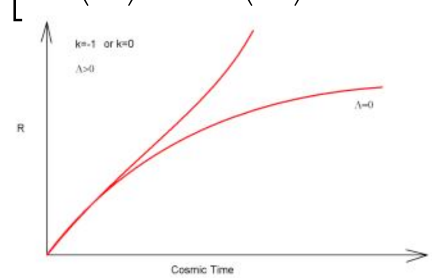
- Deceleration parameter (observable) now depends on both matter content and Λ (will discuss more later)
- This changes the relation between evolution and geometry. Depending on value of Λ,
  - closed (k=+1) Universe could expand forever
  - flat (k=0) or hyperbolic (k=-1) Universe could recollapse

Consequences of positive Λ

- Because Λ term appears with *positive* power of *R* in Friedmann equation, effects of Λ increase with time if *R* keeps increasing
- Positive Λ can create accelerating expansion!

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0^{\ 2}R_0^{\ 2}\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \Omega_k\right]$$





Special solutions

- "de Sitter" model:
  - Case with Ω_k=0 (flat space!), Ω_M =0 (no matter!), and Λ >0
  - Then modified Friedmann equation reduces to

$$\dot{R}^2 = H^2R^2 = H_0^{\ 2}R_0^{\ 2}\Omega_\Lambda\left(\frac{R}{R_0}\right)^2 = \frac{R^2\Lambda}{3}$$

  - Hubble parameter is constant:

$$H = \frac{\dot{R}}{R} = \sqrt{\frac{\Lambda}{3}}$$

  - Expansion is exponential:

$$R = R_0 e^{Ht/t_0}$$

Special solutions

- Static model (Einstein's)
  - Solution with $\Lambda=4\pi G\rho$, $k=\Lambda R^2/c^2$
  - No expansion: *H=0*, *R=constant*
  - Closed (spherical)
  - Of historical interest only since Hubble's discovery that Universe is expanding!

Other ideas

- Steady solution:
  - Constant expansion rate
  - Matter constantly created
  - No Big Bang
  - Ruled out by existing observations:
    - Distant galaxies (seen as they were light travel time in the past) differ from modern galaxies
    - Cosmic microwave background implies earlier state with uniform hot conditions (big bang)
    - Observed deceleration parameter differs from what would be required for steady model

"asymptotic" behavior

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0^{\;2}R_0^{\;2}\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \Omega_k\right]$$

- Different terms in modified Friedmann equation are important at different times…
  - Early times $\Rightarrow$ R is small
  - Late times $\Rightarrow$ R is large
- When can curvature term be neglected?
- When can lambda term be neglected?
- When can matter term be neglected?
- How does R depend on t at early times in ***all*** solutions?
- How does R depend on t at late times in ***all*** solutions?
- What is the ultimate fate of the Universe if $\Lambda \neq 0$?

Next lecture…

- The Early Universe
  - Cosmic radiation and matter densities
  - The hot big bang
  - Fundamental particles and forces
  - Stages of evolution in the early Universe