Class 18. *N*-body Techniques, Part 1

The *N*-body Problem

- Study of the dynamics of interacting particles, usually involving mutual forces. E.g.,

Mutual Force	Application
gravity	stellar dynamics, planetesimals
QM	molecular dynamics, solid-state physics
EM	plasma physics
etc.	etc.

- Stick with gravitation for now.
- Only a few literature references available, e.g., Aarseth, Danby (Ch. 9), etc.

Generalized Newton's Laws

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \boldsymbol{\mathcal{F}}_{ij} = -\sum_{j \neq i} \frac{G m_j (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}.$$

- These are $3N$ coupled 2nd-order ODEs.
- As usual, reduce to 1st-order:

$$\begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{v}_i \text{ (velocity)}, \\ \dot{\mathbf{v}}_i &= -\sum_{j \neq i} \frac{G m_j (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}i - \mathbf{r}_j|^3} \text{ (acceleration)}. \end{aligned}$$

  - This makes $6N$ coupled 1st-order ODEs.
  - We know how to solve these!

- Key is to solve the equations <u>efficiently</u>:
  1. Solve Newton's Laws using ODE integrator.
  2. Evaluate interparticle forces $\mathbf{F}_{ij}$—several techniques.

Typical Parameters

- First, need to get a feeling for the problem...
- What are typical problem sizes?

  $N \simeq 2$: Jupiter and Sun, extrasolar planets.

  $N \simeq 9$: Solar system.

$N \simeq$ 10–100: Small stellar system.

$N \simeq$ 100–1000: Open cluster, rubble pile!

$N \simeq 10^5$–10^6: Globular cluster, planetesimals.

$N \simeq 10^7$–10^8: Cosmological volume (DM halos).

$N \simeq 10^9$: Planetary rings.

$N \simeq 10^{11}$: Galaxy.

Also have "restricted" problems where one or more "test" particles exert no gravitational forces but still feel forces due to more massive particles, e.g., Lagrange problem, comets in the Oort cloud, etc.

- What are typical timescales? ($[T] = [L]/[V]$)

  Solar system: Orbital time–evolution time (1–10^9 yrs).

  Stellar system: Relaxation time ($\sim$ 100's of crossing times).

  Globular cluster: Core collapse ($\sim$ 10's of relaxation times).

  Galaxy: 10^{10} yrs (many steps).

  Universe: 10^{10} yrs (fewer steps).

- Often to achieve steady state over many dynamical times it seems $N\tau/\delta t \sim$ constant. $\Longrightarrow$ timescale and lengthscale closely coupled.
  - E.g., crossing time for closed dynamical system.

    Virial theorem: $2K + W = 0$, $K = \frac{1}{2}M\langle v^2\rangle$, $W = -GM^2/r_g$.

    Crossing time $= [L]/[V] \simeq r_g/\langle v^2\rangle^{1/2} \simeq r_g^{3/2}/\sqrt{GM}$.
    - Typically want $\delta t \simeq \tau_D/30 = \tau_{\rm cross}/30$.
  - Another handy formula:
$$\tau_D \simeq \frac{3}{\sqrt{G\rho}}.$$
    E.g., for typical asteroid, $\rho \simeq 2$ g/cc so $\tau_D \simeq 2.3$ h. For Earth, spread out mass of Sun to 1 AU: $\rho = M_\odot/\frac{4}{3}\pi r_\oplus^3 \Longrightarrow \tau_D \simeq 1$ yr. Why? $\omega^2 r = GM/r^2 \Rightarrow 4\pi^2/\tau^2 = GM/r^3 = \frac{4}{3}\pi G\rho$. $\therefore \tau \sim 3/\sqrt{G\rho}$.

Units

- In MKS, $G = 6.7 \times 10^{-11}$, $M_\odot = 2 \times 10^{30}$, $r_\oplus = 1.5 \times 10^{11}$.
- Often want to work in scaled units to keep values close to unity.
- Typically set $G \equiv 1$.
  - For solar system, use masses in $M_\odot$, distances in AU. Then times in yr/2π and speeds in $v_\oplus = 30$ km s^{-1}.
  - For galaxies, could use masses in $10^9 M_\odot$, distances in kpc. Then times would be in $\sim$ 15 Myr and speeds in kpc/15 Myr.

Constants of motion

- If there are no outside forces/torques, Newton's Laws for a gravitating system imply:
  1. Total energy is conserved.
  2. Total angular momentum is conserved.
  3. System center of mass is either stationary in the inertial frame, or moves with constant velocity.

  – Can therefore set $\mathbf{r}_g = \mathbf{v}_g \equiv \mathbf{0}$.

***N* = 2 problem**

- Solved by Kepler, explained by Newton.
- General solution (ellipse):

$$\begin{aligned} r &= a(1 - e\cos\psi) \\ \cos\theta &= \frac{\cos\psi - e}{1 - e\cos\psi} \end{aligned}$$

  where a = semi-major axis, e = eccentricity, ψ = eccentric anomaly, and mean anomaly $\omega t = \psi - e\sin\psi$ (Kepler's equation).

- Useful facts: if r and v are <u>relative</u> coordinates of two bodies, then

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{2}\frac{m_1 m_2}{M}v^2 + \frac{1}{2}Mv_g^2 - \frac{Gm_1 m_2}{r},$$

  where $M \equiv m_1 + m_2$. Hence, since we can always set $\mathbf{v}_g \equiv \mathbf{0}$,

$$\frac{E}{\mu} = \frac{v^2}{2} - \frac{GM}{r},$$

  where $\mu \equiv m_1 m_2/M$ = reduced mass. Also have

$$E = -\frac{Gm_1 m_2}{2a},$$

  (Cf. Goldstein), so:

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{GM}.$$

  In addition, if $\mathbf{h} = \mathbf{r} \times \mathbf{v} = \mathbf{L}/\mu$ = angular momentum per unit reduced mass, then

$$e = \sqrt{1 - \frac{h^2}{aGM}}.$$

  Note $h = r_p v_p = r_a v_a$, where p and a denote periapse and apoapse, respectively, and

$$r_p = (1 - e)a,\ r_a = (1 + e)a,\ r_p + r_a = 2a.$$

  Finally,

$$\cos i = \frac{h_z}{h},$$

  where i = orbital inclination wrt $z = 0$ plane.

$N > 2$ problem

> *The orbit of any one planet depends on the combined motion of all the planets, not to mention the actions of all these on each other. To consider simultaneously all these causes of motion and to define these motions by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the forces of the entire human intellect.—Isaac Newton 1687.*

- One of the earliest N-body simulations (collision of two galaxies) used lightbulbs to compute the forces! (Cf. Holmberg 1941, *ApJ* **94**, 385.)