

**DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO**

**ASTR615 — FINAL EXAM — Tuesday, May 20, 2014  
from 1:30pm - 3:30pm in CSS0201**

**WHILE YOU ARE WAITING**

**Information About the Exam**

This exam consists of multiple questions divided into three main topics. The point value of each part (cumulative points) and of each question is listed. Multiple parts questions also show the break up into points for each part. The point total for the exam is 120. Since the exam lasts 120 minutes, apportion your time accordingly! Write your answers on the provided white paper for copiers (letter size) only on one side of the paper. Is VERY IMPORTANT that you remember to write your name on the first page and mark the top right corner of each page with a progressive number, next to the total number of pages of your solution (example: 1/8, 2/8, .... , 8/8). For wordy answers, feel free to use point form. PLEASE be sure to SKIM THE ENTIRE EXAM BEFORE STARTING, as some questions are easier than others. The exam booklets will be collected at 3:30 pm.

WHEN YOU ARE FINISHED, PLEASE WRITE “I do so pledge” ON YOUR EXAM AND SIGN IT. This is in lieu of the following statement: “I pledge on my honor that I have not given or received any unauthorized assistance on this examination.”

NOTE: there is a glossary of acronyms at the end of the exam.

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1. [35 points total] **Random Numbers and Numerical Integration**

- (a) [10 points] The line profile in absence of thermal broadening (or when thermal broadening can be neglected) is described by a Lorentzian profile:

$$p(\nu; \nu_0, \gamma) = \frac{1}{\pi} \left[ \frac{\gamma}{(\nu - \nu_0)^2 + \gamma^2} \right],$$

where  $\nu_0$  is the line frequency and  $\gamma$  the line width.

Starting from a uniform deviate, derive the expression for generating a random distribution of photon frequencies that obeys the probability distribution  $p(\nu)$ .

[Hint: the derivative of  $\arctan$  is  $(1 + x^2)^{-1}$ ]

- (b) [15 points] Consider the Trapezoidal rule of integration of a function  $f(x)$  between point  $a$  and  $b$ .
- [5 points] Explain what is the basic geometrical interpretation of the Trapezoidal rule of integration and derive the formula for the area under the trapezoid.
  - [10 points] Derive the expression for the leading error in the trapezoidal integration rule. In order to do that, integrate the polynomial expansion of  $f$ :  $f(x) = f(a) + (x-a)f' + (x-a)(x-b)f''/2 + \dots$ , where  $f' \equiv [f(b) - f(a)]/(b-a)$ , and  $f''$  is the second derivative evaluated at a point inside the interval  $(b-a)$ .
- (c) [10 points] Another method to solve an integral is to transform it into an ODE and solve the ODE with appropriate boundary condition. Show the procedure for the case of Eulerian and midpoint methods of integration of the ODE, and compare the formulae to the trapezoidal rule. Are these formulae equivalent to the trapezoidal rule?

2. [40 points total] **ODEs**

- (a) [10 points] Illustrate the procedure to integrate an ODE using the midpoint method (or Runge-Kutta 2nd order) and explain why this method is second order using the Taylor expansion of the function at the midpoint, as shown in class.
- (b) [10 points] Show the procedure to integrate numerically a function  $f(x)$  by transforming it into an ODE and this time solve the ODE using the expression derived in the previous question that uses the Taylor expansion of the function at midpoint. Compare the formula to the trapezoidal rule. Are these the same?
- (c) [20 points] Stiff ODEs and Implicit methods:
- [6 points] Explain the meaning of implicit integration method and show how can be used to solve a system of  $N$  linear equations  $\mathbf{y}' = -C\mathbf{y}$ , where  $C$  is a symmetric matrix of dimension  $N \times N$  with positive eigenvalues.
  - [7 points] A higher order implicit method can be obtained averaging explicit and implicit steps. This method is often called “Crank-Nicholson”. Show that this method is unconditionally stable, although in this case the behavior as  $h \rightarrow \infty$  is to oscillate (in bounded fashion) about  $\mathbf{y} = 0$ .
  - [7 points] Show that the “Crank-Nicholson” method is second order accurate.

3. [45 points total] **PDEs and Fluid Equations**

(a) [15 points] Consider the Hyperbolic PDE

$$\frac{\partial u}{\partial t} = v \frac{\partial u}{\partial x}.$$

- i. [2 points] Write the FTCS finite difference equation on a grid.
  - ii. [10 points] Carry out the von Neumann analysis for the amplitude of the eigenmodes  $u_j^n = \xi^n e^{ikj\Delta x}$  to derive the stability criterion.  
[Hint:  $e^{iA} = \cos A + i \sin A$ ]
  - iii. [3 points] What is modification adopted to obtain a stable scheme for the Lax method, and what is the physical interpretation of the stabilizing term?
- (b) [15 points] Write down the Euler equation describing conservation of momentum in 1D including the external force due to a constant gravitational acceleration  $\mathbf{g} = -g\mathbf{k}$  in the vertical direction  $z$ .
- i. [5 points] Express the equation both in Eulerian and Lagrangian forms.
  - ii. [5 points] Consider the steady state form of the previous equation and integrate to derive the Bernoulli equation, assuming a Polytropic equation of state for the gas  $P = K\rho^\gamma$ .
  - iii. [5 points] In everyday life there are many observations that can be successfully explained by application of Bernoulli's principle. Give two examples of application of the Bernoulli equation (or Torricelli's law) to explain every day phenomena.
- (c) [15 points] Explain qualitatively, as discussed in class, what is the physical motivation for the growth of the following well known hydrodynamical instabilities:
- i. [5 points] Rayleigh Taylor instability. Explain when it happens and why. Feel free to draw a sketch.
  - ii. [10 points] Kelvin Helmholtz instability. Explain when it happens and why. To explain the mechanism of its growth feel free to draw a sketch.

## Glossary

BVP	boundary value problem
FDE	finite difference equation
FTCS	forward-time, centered-space
IVP	initial value problem
ODE	ordinary differential equation
PDE	partial differential equation
PP	particle-particle
PM	particle-mesh
RHS	right-hand side
SPH	smoothed particle hydrodynamics

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