Partial Differential Equations

Part 1

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Classification of PDEs

- *Cf. NRiC* §19.

A PDE is simply a differential equation of more than one variable (so an ODE is a special case of a PDE). PDEs are usually classified into three types:

1. Hyperbolic (second or first order in time and space)
   - Prototype is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

   (this is the 1-D version), where v = (constant) wave speed and u = amplitude.

2 Parabolic (first order in time, second order in space)

- Prototype is the diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) \tag{2}$$

(1-D), where D = diffusion coefficient, u = amplitude.

3 Elliptic (second order in space)

- Prototype is the Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \rho \tag{3}$$

(3-D), where ρ = density (if $\rho = 0$, get Laplace equation).

- Note that (1) and (2) define *initial value problems*. If $u(x)$ (and perhaps $\partial u/\partial x$) defined at $t = t_0$, then equations define how $u(x,t)$ propagates forward in time. $\therefore$ numerical solutions of (1) and (2) give *time evolution* of u (e.g., wave amplitude).
- On the other hand, (3) defines a *boundary value problem*. Given static function ρ, find static solution u satisfying BCs. $\therefore$ numerical solution of (3) gives *space distribution* of u (e.g., gravitational potential).
- Distinction between IVPs *vs.* BVPs more important than distinction between (1) and (2). Often, IVPs are mixture of hyperbolic and parabolic.

Solving Elliptic PDEs (BVP)

- Already discussed this at length for PM codes: finite differencing yields large set of coupled algebraic equations $\Longrightarrow$ large sparse banded matrix.
- Many techniques for solving matrix:
  1. Relaxation schemes.
  2. Sparse banded matrix solvers.
  3. Fourier methods.
- Use #3 when you can, #1 or #2 otherwise.

Solving Hyperbolic PDEs (IVP)

- *NRiC* §19.1.
- Overriding concern is *stability* of algorithm.

Conservative form

- Large class of IVP can be put in “flux-conservative” form:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}, \tag{4}$$

where $\mathbf{F}$ = flux of conserved quantity. In multidimensions,

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot \mathbf{F}$$

(this is in the form of a conservation law).

For example, prototypical hyperbolic PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

(v constant) can be decomposed into two first-order equations:

$$\frac{\partial r}{\partial t} = v \frac{\partial s}{\partial x}, \qquad \frac{\partial s}{\partial t} = v \frac{\partial r}{\partial x},$$

where

$$r \equiv v \frac{\partial u}{\partial x}, \qquad s \equiv \frac{\partial u}{\partial t}.$$

(can show that these two equations do indeed combine to give the original second-order equation.) Then let

$$\mathbf{u} = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} 0 & -v \\ -v & 0 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -vs \\ -vr \end{pmatrix}.$$

Plugging these into the conservative form (4) gives the decomposed version of the PDE.

The scalar advection equation

- If we can cast our hyperbolic PDE into conservative form, then all we need to do is develop numerical solution strategies for the first-order equations, which can usually be written in the form:

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} \tag{5}$$

(v still constant). We happen to already know the analytical solution is $u = f(x - vt)$, i.e., function f displaced by vt, [a]

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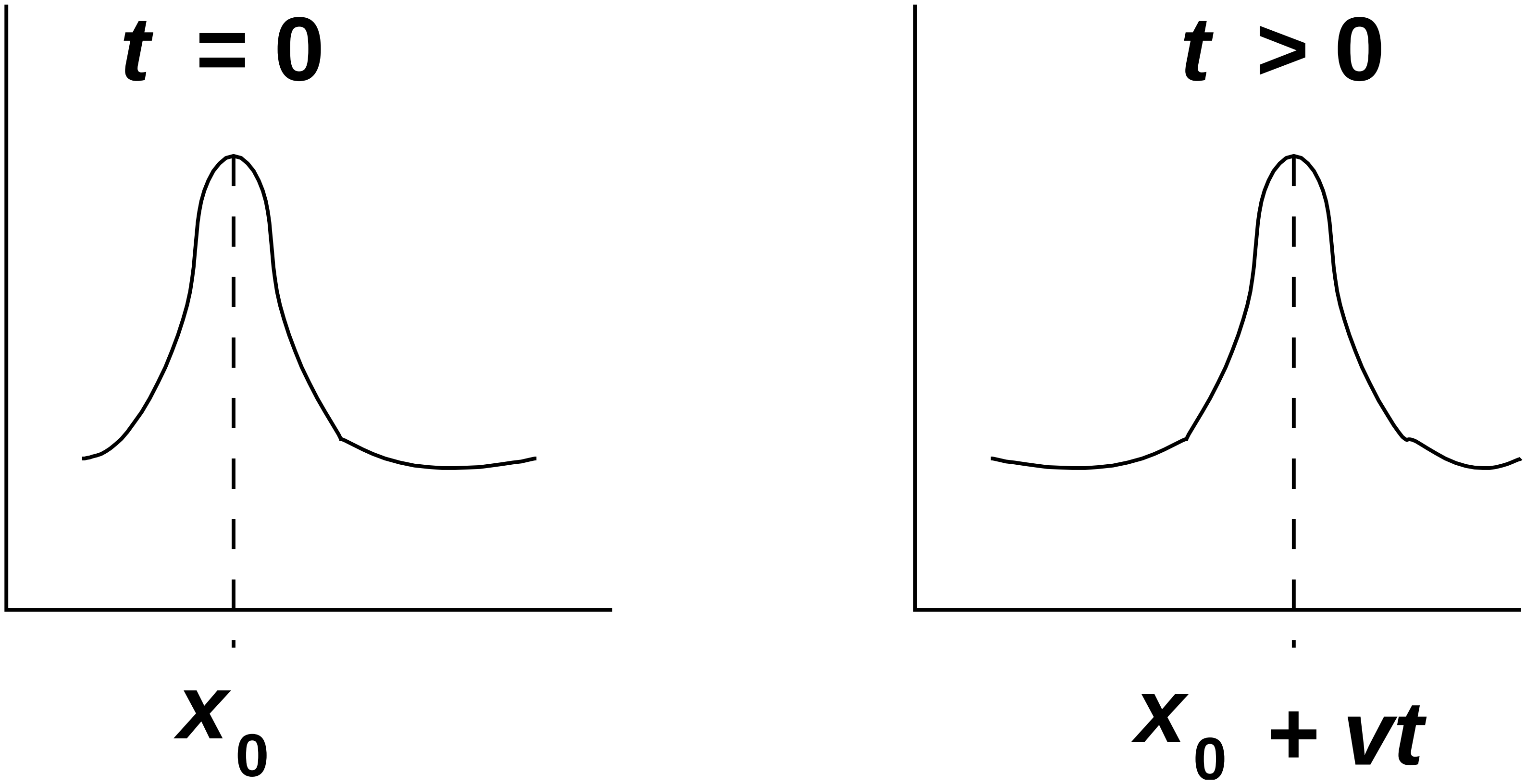
[a] To see this, let $w = x - vt$ and differentiate $u = f(w)$ using the chain rule: $\partial f/\partial t = (\partial f/\partial w)(\partial w/\partial t) = -v(\partial f/\partial w)$; $-v(\partial f/\partial x) = -v(\partial f/\partial w)(\partial w/\partial x) = -v(\partial f/\partial w)$.



but we do not necessarily know the exact form of f. Equation (5) is a scalar *advection* equation (the quantity u is transported by a “fluid flow” with a speed v).

- Best example of (5) in astrophysics is continuity equation, i.e., conservation law for some quantity with density ρ. Evolution of ρ (in 1-D) obeys

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0$$

if $\int \rho dx$ = constant, i.e., material conserved. Describes how material is mixed in ISM, how mass is transported. One of the equations of fluid dynamics.

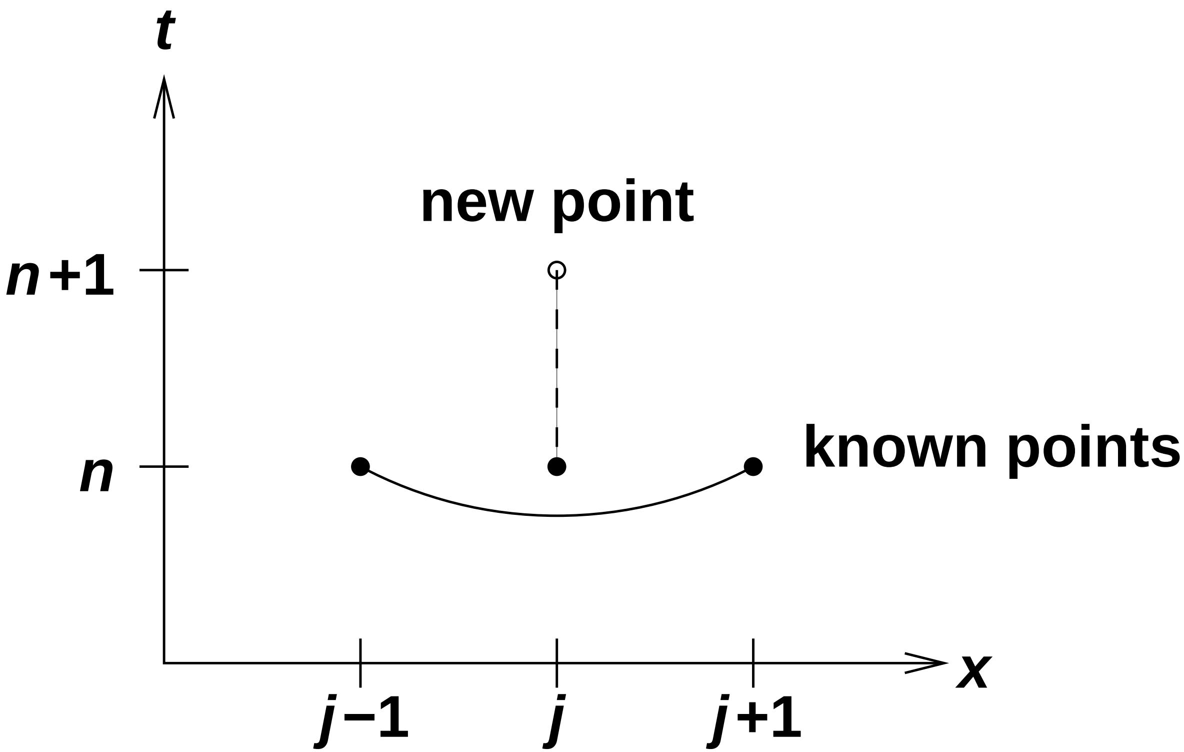
Forward time centered space scheme

- How can we construct a numerical solution to (5)?
- Try simple Euler differencing:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right). \tag{6}$$

This is first order in time and second order in space. Leads to the *forward time centered space* (FTCS) scheme.

Schematically:



- Explicit in time (just solve for u_j^{n+1}).
- What about stability of scheme?

von Neumann stability analysis

- To check stability, customary to perform a *von Neumann stability analysis*.
- Treat all coefficients of difference equations as constant in x and t (local analysis).
- Then, eigenmodes of difference equations all of form

$$u_j^n = \xi^n e^{ikj\Delta x}, \tag{7}$$

  where $\xi(k)$ is the (complex) amplitude. [a]

- The point is that the t dependence of u_j is just ξ raised to the n^{th} power. So if $|\xi(k)| > 1$ for *some* k, scheme is unstable. ξ is called the amplification factor.

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[a] Formally, the eigenmodes can be obtained from Fourier analysis of the finite-difference equations, but this is beyond our scope.

- Substitute (7) into (6), divide by ξ^n, get:

$$\xi(k) = 1 - i\frac{v\Delta t}{\Delta x}\sin k\Delta x.$$

Note $|\xi(k)| > 1$ *for all* k. $\therefore$ FTCS is *unconditionally unstable*. Too bad. Simple scheme gives garbage.

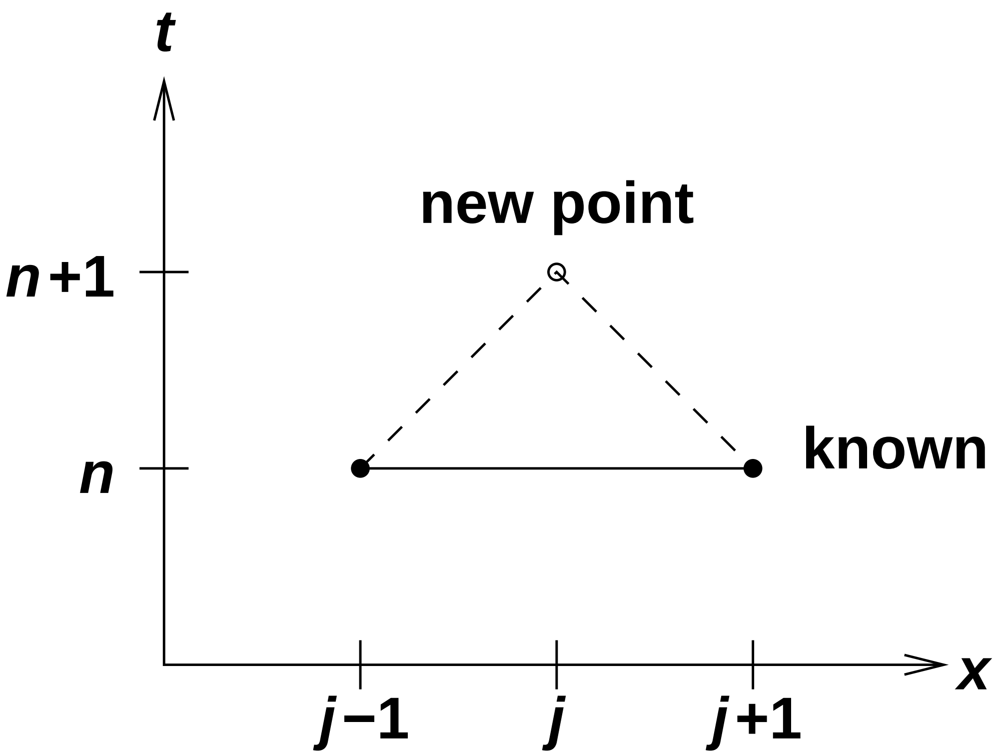
Lax scheme

- How do we fix it?
- Replace forward Euler time derivative:

$$\frac{\partial u}{\partial t} \to \frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)}{\Delta t},$$

where we have substituted the average value of u_{j-1}^n and u_{j+1}^n for u_j^n.

- Schematically:



- FDE becomes

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{v\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n), \qquad (8)$$

called the *Lax* scheme.

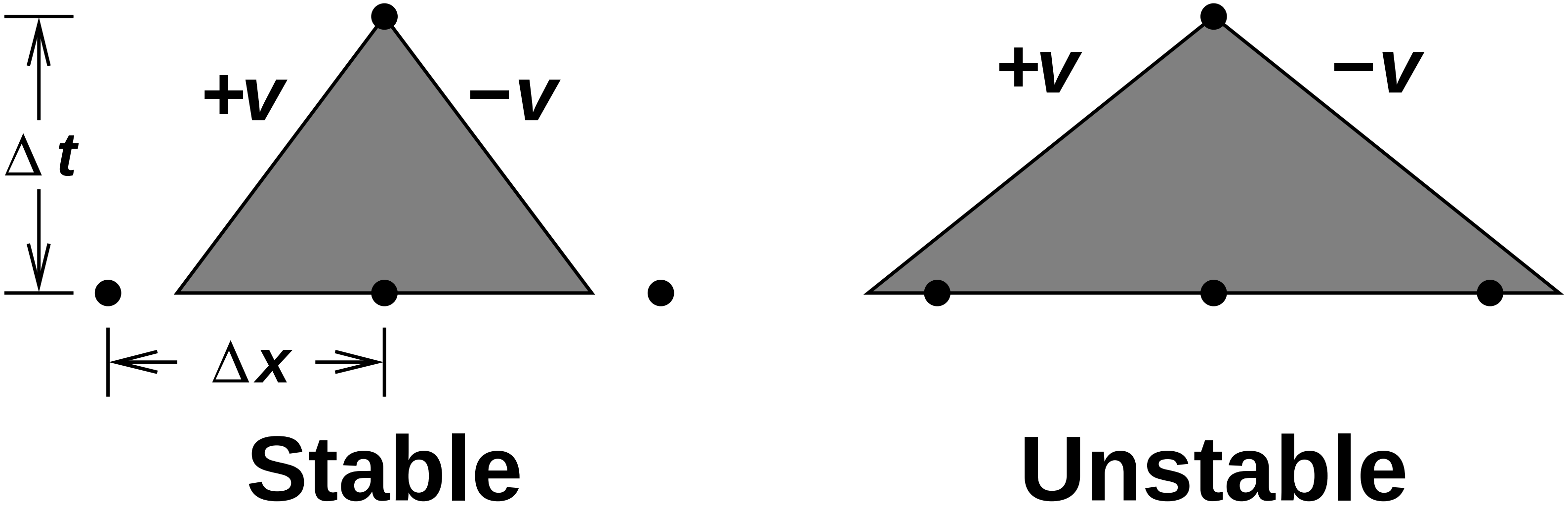
- von Neumann stability analysis of (8) gives

$$\xi(k) = \cos k\Delta x - i\frac{v\Delta t}{\Delta x}\sin k\Delta x,$$

which, for $|\xi(k)| \leq 1$, requires

$$\frac{|v|\Delta t}{\Delta x} \leq 1. \tag{9}$$

- Equation (9) is the *Courant condition* (or CFL condition, for Courant-Friedrichs-Lewy).
- Intuitively, the Courant condition can be thought of as limiting domain over which information can propagate in one timestep to be less than one gridzone, i.e., $\Delta x \geq |v|\Delta t$:



+v
−v
Δ t
Δx
Stable
+v
−v
Unstable

- Simple change in t derivative makes FTCS stable. Why? Write (8) in form of (6) with remainder term:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + \frac{1}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta t}\right).$$

But this is just FTCS representation of

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} + \underbrace{\frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion term}} .$$

- Adding diffusion stabilizes scheme: diffusion damps short wavelengths ($k\Delta x \sim 1$), leaves large wavelengths unaffected. This is called *numerical dissipation* or *numerical viscosity*.
- Damping short scales not as bad as instability!