Fluid Dynamics

Part 2

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Schemes Based on Flux-conservative Form

- By their very nature, the fluid equations (1)–(3) can be written in flux-conservative form. In 1-D, with no external forces,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = \mathbf{0},$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \\ e \end{pmatrix} \qquad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (e+p)v \end{pmatrix}$$

(E.F.T.S.). Recall $e \equiv \rho(\varepsilon + \frac{1}{2}v^2)$ = energy density.

- When written in this form, it is clear that the techniques described before can be applied immediately (given an equation of state for p).
- E.g., two-step Lax-Wendroff:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right),$$

where

$$F_{j\pm 1/2}^{n+1/2} = F(u_{j\pm 1/2}^{n+1/2}),$$

etc.

- Note that all components of $\mathbf{u}$ must be at same location on mesh $\Longrightarrow$ staggered mesh not needed (compare with operator split method, below).
- Scheme is stable provided $(|\mathbf{v}| + c)\Delta t/\Delta x < 1$, where $c^2 = \gamma p/\rho$ (c = sound speed).

Operator Split Schemes

- Simplest schemes, developed long ago by von Neumann, Richtmeyer, etc.
- The fluid equations (1)–(3) can be written as “sum” of two steps. In 1-D:

A

$$\partial\rho/\partial t + v\,\partial\rho/\partial x = 0$$

$$\partial(\rho v)/\partial t + v\,\partial(\rho v)/\partial x = 0$$

$$\partial e/\partial t + v\,\partial e/\partial x = 0$$

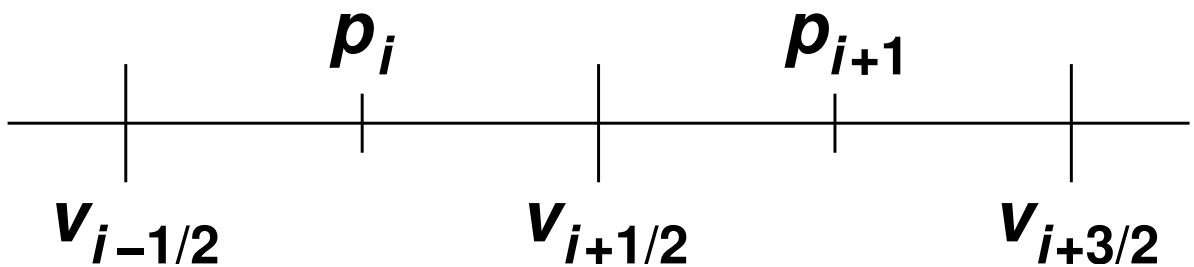
B

$$\partial\rho/\partial t = 0$$

$$\rho\,\partial v/\partial t = -\partial p/\partial x$$

$$\partial e/\partial t = -p\,\partial v/\partial x$$

- Equations in **A** are all the form of scalar advection equations. Adopt high-order upwind schemes to solve **A** $\Longrightarrow$ *advection step* (transport).
- Equations in **B** are all source terms in the equations that can be differenced directly $\Longrightarrow$ *source step*.
- Best to adopt staggered mesh with v at cell edges, p, ρ, e at cell centers.



(In multi-D, $\mathbf{v}$ defined at cell faces—naturally describes flux of fluid into/out of cell.)

- Leads to $2^{\rm nd}$-order accurate FDEs for the source terms. I.e.,

$$
\begin{aligned}
v_{i-1/2}^{n+1} &= v_{i-1/2}^{n} - \frac{\Delta t(p_i - p_{i-1})}{\frac{1}{2}(\rho_i + \rho_{i-1})\Delta x}, \\
e_i^{n+1} &= e_i^n - \frac{\Delta t(v_{i+1/2} - v_{i-1/2})p_i}{\Delta x}.
\end{aligned}
$$

- Operator split schemes are simple, easy to code, and easy to extend with more complex physics, e.g., MHD, radiation, etc.
- But, they don't treat regions with sharp p gradients as well as more modern schemes (e.g., Godunov).

Smoothed Particle Hydrodynamics

- So far we have only considered methods that require dividing space into a grid. Can we represent the local fluid density without a grid?
  - Advantage: not confined to a specific geometry, more adaptable.
- Strategy (cf. Monaghan 1992, *ARAA* **30**, 543):
  - Represent fluid by large number ($\sim 10^{4–6}$) of "particles."
  - Each particle has a mass, Lagrangian position and velocity, internal energy, and possibly an initial density.
  - Other quantities derived by smoothing over an "interpolating kernel" W (units: 1/volume). In this way the statistical properties of the real fluid elements are treated in an average sense.
  - Solve fluid equations in comoving frame with these smoothed quantities using any familiar method (leapfrog, RK, etc.)

The interpolating kernel

- If f is some quantity (e.g., density), then its kernel estimate (per unit volume) $\langle f \rangle$ is given by

$$\langle f(\mathbf{r}) \rangle = \int f(\mathbf{r}') W(\mathbf{r} - \mathbf{r}'; h)\, d\mathbf{r}',$$

where the integral is over all space,

$$\int W(\mathbf{r} - \mathbf{r}'; h)\, d\mathbf{r}' = 1,$$

and

$$\lim_{h \to 0} W(\mathbf{r} - \mathbf{r}'; h) = \delta(\mathbf{r} - \mathbf{r}').$$

(h is called the "smoothing length" and is typically chosen so that $N \simeq 15$ particles lie within h of any particle. The error for using this approximation goes as $\sim \mathcal{O}(h^2)$.)

- For numerical work, split the fluid into small volume elements $\Delta\tau$ of mass $\rho\Delta\tau$, where ρ is a representative density for the small fluid element. The integral can then be approximated by

$$\langle f(\mathbf{r})\rangle = \sum_j m_j \frac{f(\mathbf{r}_j)}{\rho(\mathbf{r}_j)} W(\mathbf{r}-\mathbf{r}_j; h).$$

  (Note m_j/ρ_j takes the place of $\int d\mathbf{r}'$.)

- The interpolating kernel can be any analytically differentiable function that satisfies the normalization and limiting properties above. E.g.,

  3-D Gaussian kernel:

$$W(\mathbf{r}; h) = \frac{1}{(\pi h^2)^{3/2}} e^{-r^2/h^2}.$$

- 3-D spline kernel:

$$W(\mathbf{r};h) = \frac{1}{\pi h^3} \begin{cases} 1 - \frac{3}{2}s^2 + \frac{3}{4}s^3 & , \ 0 \leq s < 1 \\ \frac{1}{4}(2-s)^3 & , \ 1 \leq s < 2 \\ 0 & , \ s \geq 2 \end{cases} ,$$

where $s \equiv r/h$.

In practice, choose W so that it falls off rapidly for $|\mathbf{r} - \mathbf{r}_j| \geq h$, hence only need to sum over nearest neighbours.

- Note the kernel estimate of the gradient of $f(\mathbf{r})$ is just

$$\langle \boldsymbol{\nabla} f(\mathbf{r}) \rangle = \sum_j m_j \frac{f(\mathbf{r}_j)}{\rho(\mathbf{r}_j)} \boldsymbol{\nabla} W(\mathbf{r} - \mathbf{r}_j; h)$$

(since the $\boldsymbol{\nabla}$ operator is taken with respect to the space coordinates $\mathbf{r}$ and W is the only quantity that depends on $\mathbf{r}$). However, it is often better to use the identity $\rho \boldsymbol{\nabla} f = \boldsymbol{\nabla}(\rho f) - f \boldsymbol{\nabla} \rho$ to give [a]

$$\langle \rho_i \boldsymbol{\nabla} f_i \rangle = \sum_j m_j (f_j - f_i) \boldsymbol{\nabla}_i W_{ij},$$

where $\boldsymbol{\nabla}_i W_{ij}$ is the gradient of $W(\mathbf{r}_i - \mathbf{r}_j; h)$ w.r.t. the coordinates of particle i.

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[a] Because $\langle \boldsymbol{\nabla}_i (\rho_i f_i) \rangle = \sum_j m_j f_j \boldsymbol{\nabla}_i W_{ij}$ and $f_i \langle \boldsymbol{\nabla}_i \rho_i \rangle = \sum_j m_j f_i \boldsymbol{\nabla}_i W_{ij}$.

The equations of fluid dynamics, SPH version

- Recall the continuity equation

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \mathbf{v},$$

  where the derivative is taken in the fluid frame.

- The SPH equivalent would be

$$\frac{D\rho_i}{Dt} = \sum_j m_j \mathbf{v}_{ij} \cdot \boldsymbol{\nabla}_i W_{ij},$$

  where $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$ (we used that vector identity again, replacing the gradient with the divergence).

- But we could estimate the density everywhere directly by

$$\langle \rho(\mathbf{r}) \rangle = \sum_j m_j W(\mathbf{r} - \mathbf{r}_j; h).$$

  This means we don't really need to solve the continuity equation (except in practice it turns out it *is* better to solve the equation for technical reasons; see Monaghan 1992).

- The momentum equation without external forces

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

  becomes

$$\frac{D\mathbf{v}_i}{Dt} = -\sum_j m_j \left(\frac{p_j}{\rho_j^2} + \frac{p_i}{\rho_i^2} \right) \boldsymbol{\nabla}_i W_{ij},$$

- where the pressure gradient has been symmetrized by

$$\frac{\boldsymbol{\nabla} p}{\rho} = \boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \frac{p}{\rho^2}\boldsymbol{\nabla}\rho$$

in order to ensure conservation of linear and angular momentum (the momentum equation becomes a central force law between particles i and j, assuming W is Gaussian).

- Finally, the energy equation

$$\frac{D\varepsilon}{Dt} = -\frac{p}{\rho}(\boldsymbol{\nabla}\cdot\mathbf{v})$$

becomes

$$\frac{D\varepsilon}{dt} = \frac{1}{2}\sum_j m_j \left(\frac{p_j}{\rho_j^2} + \frac{p_i}{\rho_i^2}\right) \mathbf{v}_{ij}\cdot\boldsymbol{\nabla}_i W_{ij},$$

where the factor of $1/2$ comes from symmetrization (it is a characteristic of SPH that gradient terms can be written in many different ways, just as there are a variety of FDE representations).

- Particles are moved using

$$\frac{D\mathbf{r}_i}{Dt} = \mathbf{v}_i,$$

or, to ensure particles move with a velocity similar to the average velocity in their neighbourhood,

$$\frac{D\mathbf{r}_i}{Dt} = \mathbf{v}_i + \epsilon \sum_j m_j \left(\frac{\mathbf{v}_{ji}}{\bar{\rho}_{ij}} \right) W_{ij},$$

where $\bar{\rho}_{ij} \equiv (\rho_i + \rho_j)/2$ and $0 \leq \epsilon \leq 1$ is a constant (the "X" factor).

- As usual, also need an equation of state.
- Can add other forces, i.e., viscosity, magnetic fields, etc.
- Can implement adaptive smoothing lengths.

Summary

- SPH is based on microscopic picture of real fluid. But real fluid has many more particles than can be followed on a computer, so "smoothing" is used.
- Advantage of scheme is that it is adaptive—particles go where density is high. Good for following dynamics where gravity dominates because of its N-body-like foundation.
- Disadvantage of scheme is that it does not resolve low-density regions well, it does not handle regions with strong p gradients well (shocks), and it is expensive (need a way to find nearest neighbours $\Longrightarrow$ tree code!).
- Finite differencing methods are attractive because mathematical properties of FDEs well studied, and can prove/analyze stability, convergence rate, etc. of various schemes.