Cosmological Particle Creation: A Quantum Field Theoretic Perspective

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May 12, 2008

1 Introduction

The advent of quantum field theory (QFT) was marked by a the need for a new quantum theory of matter. Ordinary quantum mechanics is well suited to handle problems with few particles at low energies. But, much interesting physics happens at high energy where particle creation occurs and where relativistic effects become important. QFT represents the successful marriage of quantum theory with relativity and is equipped to handle particle creation. It is the particle creation aspect of QFT that we wish to explore here.

2 Canonical Quantization

To ease the presentation we cover some basic aspects of QFT that are important for the following arguments. To begin we ask a simple question, "How do we do QFT?". To answer this question we explore the simplest approach, that of canonical quantization. The structure is simple, we begin with the Lagrangian for the field of interest, and in our case we choose a massive scalar field.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1}$$

Employing the methods of classical physics we can find the conjugate momenta (to the field) by computing the variational derivative with respect to $\dot{\phi}$.

$$\Pi = \frac{\delta S}{\delta\dot{\phi}} \tag{2}$$

With the definition of the canonical momentum we can find the Hamiltonian for the field via the the Legendre transform of the Lagrangian.

$$H = \int d^3x[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2] \tag{3}$$

To quantize the theory we promote the field and canonical momentum to quantum operators and impose equal time commutation relations.

$$[\hat{\phi}(\vec{x},t),\hat{\Pi}(\vec{x}',t)] = i\hbar\delta(\vec{x}-\vec{x}') \tag{4}$$

$$[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{x}',t)] = 0 \tag{5}$$

$$[\hat{\Pi}(\vec{x},t),\hat{\Pi}(\vec{x}',t)] = 0 \tag{6}$$

After imposition of the equal time commutations relations we can now use our Hamiltonian (operator valued) to find the Heisenberg equations of motion.

$$\frac{d\hat{\phi}}{dt} = \frac{1}{i\hbar}[\hat{\phi}(x),\hat{H}] \tag{7}$$

This leads to the field equation, which is the same equation that can be obtained by variation of the action, S, with respect to the field, we get.

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{8}$$

This particular form of our quantum field theory is not the most useful for describing creation and destruction of particles. The "second quantized" form most easily describes particle creation. We arrive at the second quantized form by first solving the classical field equation and expressing the field in terms of orthonormal modes.

$$\phi(x) = \int d\mu(\vec{k})[a_k u_k(x) + a^\dagger u_k^*(x)] \tag{9}$$

Where the inner product is defined as,

$$(\phi_1,\phi_2) = -i\int_\Sigma d\Sigma^\mu \sqrt{-g_\Sigma}\phi_1(x)\overleftrightarrow{\partial}_\mu\phi_2(x) \tag{10}$$

Σ is a future directed spacelike hypersurface, in mikowski space often taken to be a $t = const$ surface. g_Σ is the determinant of the induced metric. The mode solutions to the field equation satisfy

$$(u_k, u_k') = \delta_{kk'} \tag{11}$$

$$({u_k}^*, {u_k'}^*) = -\delta_{kk'} \tag{12}$$

$$(u_k{}^*, u'_k) = 0 \tag{13}$$

To continue we promote the coefficients of the mode decomposition to quantum operators. It can be shown that the equal time commutation relations imply that,

$$[a_k, a^\dagger{}_{k'}] = \delta_{kk'} \tag{14}$$

$$[a_k, a_{k'}] = 0 \tag{15}$$

and that the Hamiltonian takes the form

$$\hat{H} = \int d\mu(\vec{k})\hbar\omega[a^\dagger a + 1/2] \tag{16}$$

where $d\mu$ is an appropriate measure for the mode decomposition. The interpretation now becomes obvious, the energy i.e. $\langle H \rangle$ and commutation relations are those of the creation and annihilation operators for a simple harmonic oscillator with circular frequency ω. Where special attention is taken to match positive frequency modes with annihilation operators i.e. modes that satisfy the eigenvalue equation $\mathcal{L}u_k = -i\omega u_k$, where $\mathcal{L}$ is the Lie derivative along a timelike killing vector. All properties of these operators easily follow from the commutation relations. Particularly, we can define the Fock space,

$$a_k\,|0\rangle = 0 \quad \forall k \tag{17}$$

$$a_k\,|n\rangle = \sqrt{n_k}\,|n_k - 1\rangle \tag{18}$$

$$a_k^\dagger\,|n_k\rangle = \sqrt{n_k + 1}\,|n_k + 1\rangle \tag{19}$$

where n_k is the number of quanta in the kth mode. The main point to take from this section is that it is the coefficients of the mode decomposition that become the quantum operators, and these coefficients define the Fock space.

3 Quantum Field Theory in Curved Space

To do QFT in curved space we need to account for the effects of curvature. All derivatives are replaced by covariant derivatives, the coordinate volume element is multiplied by the fundamental volume element, $\sqrt{-g}$, and we must add all local interactions with the curvature with the correct dimensions. For a massive scalar field in curved space the action takes the form

$$S = \frac{1}{2}\int d^n x\sqrt{-g}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + \xi R\phi^2] \tag{20}$$

where only the partial derivative plays a role in the action because the derivative acts on a scalar, and R is the scalar curvature. In such a model we are not accounting for the backreaction of the quantum field on the spacetime, we specify the geometry of the space and study the effects curvature has on the quantum field.

In this section we skip all the formalism of canonical quantization. We vary the action with respect to the field to yield the classical field equation, we solve the field equation to find the mode solutions, and promote the coefficients in the mode expansion to quantum operators.

$$\nabla_\mu \partial^\mu \phi + (m^2 + \xi R)\phi = 0 \tag{21}$$

For general spaces there are often many mode solutions to choose from, and often positive frequency modes of one solution have nonvanishing innerproduct with negative frequency modes of another solution. Consider two solutions u_{1k} and u_{2k}. We can write the field operator as,

$$\phi = \sum_k [a_k u_{1k} + h.c.] \tag{22}$$

or equivalently as

$$\phi = \sum_k [b_k u_{2k} + h.c.] \tag{23}$$

where a is the destruction operator for mode solution 1, b is the destruction operator for mode solution 2, and the sum on k represents the appropriate mode sum. Both sets of modes form a complete set, and thus we can represent a mode function of one solution in terms of the other.

$$u_{1i} = \sum_j [\alpha_{ij} u_{2j} + \beta_{ij} u_{2j}] \tag{24}$$

Using this and the previous two relations we can write the annihilation operators for one solution in terms of creation *and* destruction operators for the other. This yields the so called Bogoliubov transformations.

$$a_k = \sum_j [\alpha_{kj} b_j + \beta^*_{kj} b^\dagger_j] \tag{25}$$

$$b_k = \sum_j [\alpha^*_{kj} a_j - \beta^*_{kj} a^\dagger_j] \tag{26}$$

These relations tell us that for general spaces any two sets of modes do not necessarily share the same vacuum i.e. $a_k |0_2\rangle \neq 0$. This failure of two modes to have the same vacuum is encoded in the coefficient β. This is at first very troubling. Have we lost our prized concept, particles? The answer is yes. The notion of particles is defined given the global properties of the space i.e. the mode structure. For example, nonstatic solutions to the einstein equation will not possess a timelike killing vector, without which we have no way to define positive frequency modes. This is not just a feature of doing QFT in curved spaces, even in flat spacetime a uniformly accelerated detector in vacuum will detect a thermal spectrum of particles with temperature proportional to the acceleration [1]. But, in certain situations we can make fruitful analysis regarding particles, for example asymptotically flat spacetimes.

4 Example

In this section we employ the formalism of the previous sections to work out an explicit example of cosmological particle creation. Our example is a massive scalar field minimally coupled i.e. $\xi = 0$ in an expanding universe in (1+1), where all spaces are conformal to minkowski space. We follow the treatment of Birrell and Davies [2]. We write our metic

$$ds^2 = C(\eta)[d\eta^2 - dx^2] \tag{27}$$

where $\eta = \int dt/a(t)$, and $C(\eta) = a^2(\eta)$. We pick $C(\eta) = A + B\tanh(\rho\eta)$. Now we wish to find the mode solutions. The spatial part of the metric is still homogeneous so we write solutions for the modes as

$$u_k = \frac{1}{\sqrt{2\pi}}\chi_k(\eta)e^{ikx} \tag{28}$$

From the field equation we find that χ satisfies

$$\ddot{\chi} + [k^2 + m^2C(\eta)]\chi_k = 0 \tag{29}$$

For this model we can find many mode solutions, the two that are relevant to our presentation here are

$$u_1 = \frac{1}{\sqrt{4\pi\omega_{in}}}\exp\{ikx - i\omega_+\eta - (i\omega_-/\rho)\ln[2\cosh(\rho\eta)]\} \tag{30}$$
$$\times{}_2F_1(1 + (i\omega_-/\rho), i\omega_-/\rho : 1 - (i\omega_{in}); (1 + \tanh(\rho\eta))/2)$$

$$u_2 = \frac{1}{\sqrt{4\pi\omega_{out}}}\exp\{ikx - i\omega_+\eta - (i\omega_-/\rho)\ln[2\cosh(\rho\eta)]\} \tag{31}$$
$$\times{}_2F_1(1 + (i\omega_-/\rho), i\omega_-/\rho; 1 + (i\omega_{out}); (1 - \tanh(\rho\eta))/2)$$

where

$$\omega_{in} = \sqrt{k^2 + m^2(A - B)} \tag{32}$$

$$\omega_{out} = \sqrt{k^2 + m^2(A + B)} \tag{33}$$

$$\omega_\pm = \frac{1}{2}(\omega_{out} \pm \omega_{in}). \tag{34}$$

We see that u_1 reduces to standard positive frequency plane waves in the asymptotic past, we call these the 'in-modes',

$$u_1 \sim e^{ikx - i\omega_{in}\eta} \tag{35}$$

and in the asymptotic future we have that u_2 reduces to the standard positive frequency plane wave form, these are the 'out-modes'.

$$u_2 \sim e^{ikx - i\omega_{out}\eta} \tag{36}$$

Thus the physical vacuum is defined by the u_1 in the asymptotic past and by u_2 in the future. Because these two sets of modes are not the same we find a nonvanishing coefficient β when we relate in-modes to out-modes.

$$\beta_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1 - (i\omega_{in}/\rho))\Gamma(i\omega_{out}/\rho)}{\Gamma(1 + (i\omega_{-}/\rho))\Gamma(i\omega_{-}/\rho)} \tag{37}$$

There is only one index because the transformation is diagonal.

Now let us interpret this result. Let our initial state be the vacuum defined by the in-modes, $|0_1\rangle$. If we work in the Heisenberg picture our state is constant in time, but as our space evolves we find that $|0_1\rangle$ is no longer regarded as the physical vacuum in the asymptotic future, the vacuum in the future is defined with respect to u_2, and thus inertial observers in the asymptotic future will detect particles in the state $|0_1\rangle$. If we compute the number of particles in the nth mode in this state at late times we find,

$$N_k = \langle 0_1| b_k^\dagger b_k |0_1\rangle = |\beta_k|^2 = \frac{\sinh^2(\pi\omega_{-}/\rho)}{\sinh(\pi\omega_{in}/\rho)\sinh(\pi\omega_{out}/\rho)} \tag{38}$$

where it should be noted that the appropriate number operator is defined in terms of the creation and annihilation operators corresponding with the out-modes.

By analyzing our expression for the particle production we find that $|\beta_k| \sim e^{-2\pi\omega_{in}/\rho}$. From this we infer the expected result that the particle production is exponentially suppressed at high energies.

5 Conclusion

We have seen that through the practice of standard quantum field theory in curved space that the notion of particles is an observer dependent concept, and that in general the vacuum is not unique. We used a simple example with a well defined notion of particles i.e. asymptotically flat, to study the general aspects of cosmological particle production. As expected we see that the production of high energy quanta is suppressed.

[1] Unruh, W. G. (1976) Phys. Rev. D, 10, 3194

[2] Birrel and Davies in *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982), Chap. 3.