Class 6  
General Relativistic Cosmology (II)

- This class...
  - Recap the CP-compliant geometries
  - Finish discussion of metrics
  - The FRW metric
  - Photon propagation in FRW metric
    - Redshift - scale factor connection
    - Redshift – cosmic time connection
    - The particle horizon

O : Recap – possible geometries of the Universe

- The Cosmological Principle: there exists an observer who sees "simultaneous spatial slices" that are always isotropic and homogeneous... this powerful restriction limits the spatial geometry to one of three cases:
  - Spherical (positive curvature)
  - Flat
  - Hyperbolic (negative curvature)
I : The Friedmann-Robertson-Walker (FRW) Metric

- What is the metric for these spacetimes? It can be shown that the appropriate metric is...

\[ ds^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

- Notes:
  - This covers all three geometries
    - k>0 : Spherical
    - k=0 : Flat
    - k<0 : Hyperbolic
  - r, θ, φ are co-moving coordinates
  - Role of a(t) as scale factor is made clear
  - Substituting this metric into Einstein’s Field Equations gives the Friedmann equation and the acceleration equation that we’ve already encountered (thus also encapsulates the fluid equation)

II : Photon propagation in FRW metric

- In General (and Special) Relativity, photons follow paths such that the space-time interval is zero, \( ds^2 = 0 \). These are called *null geodesics*.

- Recall the FRW metric

\[ ds^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

- Consider a photon propagating in the r-direction. Put \( ds^2 = 0 \) and re-arrange to get

\[ \frac{c \ dt}{a} = -\frac{dr}{\sqrt{1 - kr^2}} \]

\[ \int_{t_e}^{t_f} \frac{c \ dt}{a} = \int_0^{r_0} \frac{dr}{\sqrt{1 - kr^2}} \]

- Here, \( t_e \) is time photon is emitted, \( t_r \) is time photon is received
Redshift – a(t) connection

- A fundamental observable in cosmology is the “redshift” defined as:

\[ z = \frac{\lambda_0 - \lambda_e}{\lambda_e} \Rightarrow 1 + z = \frac{\lambda_0}{\lambda_e} \]

- From result on previous page, we can prove a very important result (see board)...

\[ 1 + z = \frac{a(t_r)}{a(t_e)} \]

- This directly relates changes in scale factor to measured redshift
- Same statement as saying that λ stretches in proportion to scale factor...we’ve already seen this result before (when discussing fluid equation)

III : Size of the Observable Universe (The Particle Horizon)

- Suppose a photon is emitted at the time of the big bang
- How far will it have propagated in time t?
- For simplicity, let’s think about a flat matter dominated Universe. So we can put \( k=0 \) and \( a=(t/t_0)^{2/3} \) to get

\[ r_0 = 3ct_0 \]

- Notes:
  - Despite fact that \( da/dt \to \infty \) as one approaches big bang, photons emitted at big bang travel finite distance. True for all geometries.
  - \( r_0 \) is larger than \( ct_0 \)...photons cover more ground (in terms of coordinate r) when scale factor is smaller
  - \( r_0 \) defines the **size of the observable Universe**...it is impossible to obtain information from any further than this distance since no signal could have traveled further within the age of the Universe