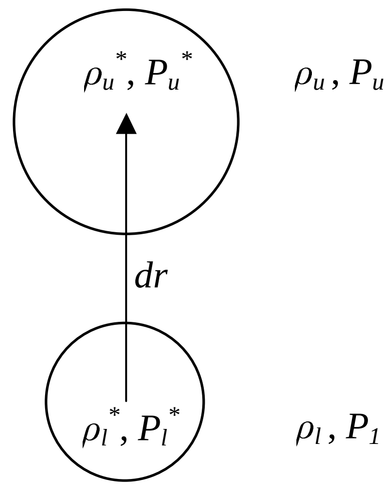
# ASTR 498N  Lecture 12  Convection

(online at www.astro.umd.edu/~drabin/)

*Double, double toil and trouble,*
*Fire burn and cauldron bubble.*
W. Shakespeare

**Convective instability**

Consider a perturbation in which a mass element in a star, initially indistinguishable from its surroundings, is displaced infinitesimally upward by *dr*. Assume that the rise is fast enough that no heat is exchanged with the surrounding gas—i.e., the perturbation is adiabatic—but slow enough that pressure equilibrium is maintained at all times.




If $\rho_u^* > \rho_u$, the risen element will find itself more dense than its surroundings and it will sink back down. This is the criterion for stability against convection.

Because of pressure equilibrium, we can drop the * on pressure quantities. Also, the initial condition requires $\rho_l^* = \rho_l$.

Carry out Taylor expansions for the density outside and inside the bubble:

$$\rho_u = \rho_l + \left(\frac{d\rho}{dr}\right)_l dr \qquad\qquad \rho_u^* = \rho_l^* + \left(\frac{d\rho^*}{dr}\right)_l dr$$

For the rising bubble, we use the *P*-and-*V* (or $\rho$) adiabatic equation:

$$\frac{d\rho_l^*}{\rho_l^*} = \frac{1}{\Gamma_1}\frac{dP_l^*}{P_l^*} \quad \text{or} \quad \left(\frac{d\rho^*}{dr}\right)_l = \frac{\rho_l}{\Gamma_1 P_l}\frac{dP_l}{P_l}$$

where we have dropped the * on pressure and used $\rho_l^* = \rho_l$. Substituting this back in the equation for $\rho_2^*$ gives

$$\rho_u^* = \rho_l + \frac{1}{\Gamma_1}\frac{\rho_l}{P_l}\left(\frac{dP}{dr}\right)_l dr$$

Comparing this with the equation

$$\rho_u = \rho_l + \left(\frac{d\rho}{dr}\right)_l dr$$

for $\rho_u$ we find that $\rho_u^* > \rho_u$ if and only if

$$\boxed{\frac{1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr} > \frac{d\rho}{dr} \quad \text{or} \quad \frac{d\ln P}{d\ln\rho} > \Gamma_1 = \left(\frac{d\ln P}{d\ln\rho}\right)_{\text{ad}} \quad \text{for convective stability}}$$

where, in this final step, we refer everything to the unperturbed level and drop the subscript *l*.

We may express this fundamental result in another form by using the general equation form of the equation of state derived in the last lecture,

$$\frac{d\rho}{\rho} = -\alpha T\frac{dT}{T} + \kappa P\frac{dP}{P}$$

to eliminate $d\rho$ in favor of $dT$ and $dP$:

$$\frac{1}{\Gamma_1}\frac{\rho}{P}\frac{dP}{dr} > \frac{d\rho}{dr}$$

$$\frac{1}{\Gamma_1}\frac{dP}{P} > \frac{d\rho}{\rho} = -\alpha T\frac{dT}{T} + \kappa P\frac{dP}{P}$$

$$-\alpha T\frac{dT}{T} < \left(\frac{1}{\Gamma_1} - \kappa P\right)\frac{dP}{P}$$

$$\frac{dT}{T} > \left( \frac{\kappa P}{\alpha T} - \frac{1}{\Gamma_1 \alpha T} \right) \frac{dP}{P}$$

where the sense of the inequality reverses upon dividing by $-\alpha T$. Finally, use the relationships $\Gamma_1 = \gamma / \kappa P$ and $\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\gamma}{\gamma - 1} \frac{\alpha T}{\kappa P}$ (derived in the last lecture) to write

$$\frac{dT}{T} > \frac{\kappa P}{\alpha T} \left( 1 - \frac{1}{\gamma} \right) \frac{dP}{P} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{dP}{P}$$

and thus

$$\boxed{\frac{dT}{T} > \frac{\Gamma_2 - 1}{\Gamma_2} \frac{dP}{P} \quad \text{or} \quad \frac{d \ln T}{d \ln P} > \nabla_{\text{ad}} \quad \text{or} \quad \frac{dT}{dr} > \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \frac{dP}{dr}}$$

We recognize the last relationship, when in the form of an equality, as the stellar structure equation that specifies the temperature gradient when it is determined by adiabatic convection. As both *dT/dr* and *dP/dr* are negative, we also have

$$\boxed{\left| \frac{dT}{dr} \right| < \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \left| \frac{dP}{dr} \right| \quad \text{for convective stability}}$$

Our derivation identifies the correct Γ to use in this criterion (O&C treat the simplified case where all the Γ's reduce to $\gamma = C_P / C_V$ ).

**When will convection occur?**

Convection will occur when the temperature gradient predicted by radiative transport violates the convective stability criterion. If we substitute the appropriate stellar structure equations for $|dT/dr|$ and $|dP/dr|$, we get

$$\frac{3\bar{\kappa}\rho}{4acT^3} \frac{L}{4\pi r^2} > \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \frac{GM}{r^2} \rho \qquad \text{(for } \textit{in}\text{stability)}$$

Remembering that $L/4\pi r^2 = F_{\rm rad}$ and $GM/r^2 = g$ (local acceleration of gravity), we have

$$\frac{3\bar{\kappa}\rho}{4acT^3}\frac{F_{\rm rad}}{g} > \frac{\Gamma_2 - 1}{\Gamma_2}\frac{\rho T}{P}$$

For the case of an ideal gas, use $P = \rho kT/\mu m_H$ on the r.h.s. to write

$$\frac{\Gamma_2}{\Gamma_2 - 1}\frac{\bar{k}F_{\rm rad}}{\mu T^3 g} > \frac{4acm_H}{3k}$$

where we have used the mean linear opacity $\bar{k} = \bar{\kappa}\rho$. This identifies at least five conditions that promote convective instability:

1. High linear opacity, as can occur in the outer envelope of a cool star.
2. Low gravity (giants and supergiants).
3. High radiative flux, as tends to occur in the core of a star with highly temperature-sensitive nuclear burning.
4. The adiabatic exponent $\Gamma_2$ is close to unity, as occurs in ionization zones (here we adhere to conventional practice by folding the effects of partial ionization into the Γ's).
5. All other things being equal, low temperature.

**Mixing Length "Theory"**

After a convective element has risen a distance *dr*, it will have a temperature excess relative to its surroundings given by

$$dT = \left(\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right)dr = \Delta\nabla\mathrm{T}\,dr$$

which defines the symbol $\Delta\nabla\mathrm{T}$ (note that this quantity is positive, as the adiabatic gradient is less negative than the actual gradient when convection occurs). This temperature excess entails a corresponding excess of thermal energy per unit volume given by $\delta Q = c_p \rho \, \Delta\nabla T \, dr$, where $c_p$ is the specific heat per unit mass at constant pressure and, if the element is rising with speed $v$, an energy flux

$$H = c_p \rho \, \Delta\nabla T \, v \, dr$$

A sinking element carries the same energy flux, since $v$ and $dr$ both change sign.

We estimate $v$ by considering the buoyancy force on a convective element that has risen by $dr$. The density of the element differs from its surroundings by

$$d\rho = \left( \frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr} - \frac{d\rho}{dr} \right) dr$$

For the purposes of estimation, assume an ideal gas and use

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$$

to eliminate $d\rho/dr$ in the preceding equation:

$$\begin{aligned} d\rho &= \left( \frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{P} \frac{dP}{dr} + \frac{\rho}{T} \frac{dT}{dr} \right) dr \\ &= -\frac{\rho}{T} \left( \frac{\Gamma_1 - 1}{\Gamma_1} \frac{T}{P} \frac{dP}{dr} - \frac{dT}{dr} \right) dr = -\frac{\rho}{T} \Delta\nabla \mathrm{T} \, dr \end{aligned}$$

where we ignore the distinctions between the $\Gamma$'s. Note that $d\rho$ is negative, as expected for a buoyant element. By Archimedes' principle, the density-deficient bubble experiences an upward force $g|d\rho|$, where $g$ is the local gravitational acceleration. This force applies at the *end* of the displacement; at the beginning, the force is zero. So, let us estimate the average force as half the final force and multiply the average force by the displacement to estimate the final velocity:

$$\frac{1}{2} \rho v^2 = \frac{1}{2} g |d\rho| \, dr = \frac{1}{2} \frac{GM}{r^2} \frac{\rho}{T} \Delta\nabla \mathrm{T} \, dr^2$$

Because this is quadratic in both $v$ and $dr$, it holds for both rising and sinking elements.

As the final piece of the puzzle, we introduce the average vertical distance over which a rising or sinking element moves before dissolving into its surroundings and call this the *mixing length, l*. Roughly, we'll say that the average displacement $\overline{dr}$ that should be used in the kinetic energy expression is half the mixing length. We can then solve for the velocity and substitute the result in the expression for the convective heat flux to obtain

$$\boxed{H = c_p \rho \left( \frac{GM}{Tr^2} \right)^{1/2} \left( \Delta \nabla T \right)^{3/2} \frac{l^2}{4}}$$

This is the desired relationship between the convective heat flux and the temperature gradient (expressed as an excess over the adiabatic gradient).

Amidst the considerable hand-waving here, there is a particularly large waving hand, namely, the mixing length. Although mixing-length theory can be clothed in fancier dress than it is here, it has been for half a century, and remains, one of the central uncertainties in modeling stellar structure. You might think this a bit disgraceful—isn't *everything* solved after fifty years?—but it's not so surprising when you consider that real-life convection is turbulent, and turbulence is a famously difficult problem.

Nevertheless, all is not despair. From empirical experience with convection, it is reasonable to expect that the mixing length should be comparable, in order of magnitude, with the pressure scale height. As shown in O&C Example 10.7, applying that ansatz to the Sun at the base of the convection zone shows that convection can carry all the flux with a temperature gradient that exceeds the adiabatic gradient by only about one part in a million (the corresponding velocities are of order $10^{-4}$ of the sound speed). Thus, for the purpose of computing the stellar structure in the interior, we can simply adopt the adiabatic temperature gradient in convective regions, with negligible error.

Near the surface of a star, the pressure scale height (and presumptively the mixing length) is small, the density is low, the temperature is low, and the specific heat may be large because of partial ionization—all of which act to drive up $\Delta \nabla T$ and *v*, sometimes to the point that it is no longer a good approximation to use the adiabatic gradient and *v* may approach the sound speed. In these cases, hand-waving fails.

An important practical consequence of the failure of mixing length theory in the outer layers of convective stars is the inability to calculate the stellar radius with any precision. To see this, consider the base of an outer convection zone to be at a welldetermined radius and temperature (since mixing length theory is applicable in the deeper interior). The temperature needs to approach zero at the surface of the star, in the sense that the stellar effective temperature is much smaller than the temperature at the base of the convection zone. The radial distance needed to reduce the temperature that much depends on the temperature gradient. If we can't accurately determine the temperature gradient, we can't determine the distance from the base of the CZ to the surface, and hence the radius.

**Convective overshoot**

Our treatment of convection has been *local*: we predict that convection will occur at a given radius if the predicted radiative temperature gradient exceeds the adiabatic gradient. The upper and lower boundaries of the CZ are well-defined and sharp. On the other hand, we posit that convective elements rise or sink by a mixing length length and acquire upward or downward velocity before they merge with their surroundings. This is clearly not a consistent picture when we are close to a CZ boundary: there is no force to stop dead a moving convective element at the "exact" boundary; rather we expect some of the elements to *overshoot* into the stably stratified layers until the negative buoyancy they experience there slows them down and stops them in a distance comparable to the mixing length.

Overshoot has practical consequences. For example, overshooting in the convective cores of a massive star can extend its predicted lifetime (by mixing in fresh fuel) and change the surface abundances (by mixing up the products of nuclear burning). Unfortunately, because overshoot is nonlocal, it is even more difficult to treat than ordinary convection. There are several approaches in use, none generally accepted.