ASTR 498N Lecture 16 Stars That Can Be

(online at www.astro.umd.edu/~drabin/)

♫ Be all that you can be.

Minimum mass of a main sequence star

Back in Lecture 5, we made rough order-of-magnitude estimates of the central pressure and temperature of a star, based on hydrostatic support and the ideal gas law, $P = \rho k T_c / \mu m_H$:

$$P_c \approx \frac{1}{4\pi}\frac{GM^2}{R^4} = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G\bar{\rho}^{4/3}M^{2/3}$$

$$T_c \approx \frac{\mu m_H}{k}\frac{GM}{3R} = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G\mu m_H \bar{\rho}^{1/3}M^{2/3}$$

In Homework 4, you derived an upper bound on P_c related to the central density rather than the average density:

$$P_c < \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} G\rho_c^{4/3}M^{2/3}$$

Let us set the central pressure to this upper limit. For actual stars, which are centrally concentrated, this approximation is better than expressions pegged to the average density; realistic models yield central pressures that are roughly half the maximum value.

Consider the initial contraction of a protostellar cloud. Equating our approximation to the central pressure to the perfect gas pressure, we find

$$kT_c = \left(\frac{\pi}{6}\right)^{1/3} G\mu m_H \rho_c^{1/3}M^{2/3}$$

which shows how the central temperature rises with the central density.

What happens as the contraction continues? (We largely follow the presentation in *The Physics of Stars* by A. C. Phillips). Set aside for the moment the possibility of nuclear burning. When the central density becomes high enough, the pressure of degenerate, nonrelativistic electrons becomes significant. The total pressure becomes

$$P_c = K_{\mathrm{NR}} n_e^{5/3} + n_i k T_c \qquad \text{where } K_{\mathrm{NR}} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}$$

Assume for simplicity a composition of pure hydrogen, so that

$$n_e = n_i = \rho_c / m_H$$

Again equating the two expressions for central pressure,

$$kT_c = \left(\frac{\pi}{6}\right)^{1/3} G m_H \rho_c^{1/3} M^{2/3} - K_{\mathrm{NR}}\left(\frac{\rho_c}{m_H}\right)^{2/3} \equiv A\rho_c^{1/3} - B\rho_c^{2/3}$$

Clearly this expression has a maximum, shown by differentiation to have the value $A^2/4B$ at a density of $(A/2B)^3$. Substituting for *A* and *B*, we find

$$\left[kT_c\right]_{\max} = \left(\frac{\pi}{6}\right)^{2/3} \frac{G^2 m_H^{8/3}}{4K_{NR}} M^{4/3}$$

Suppose that proton-proton fusion "ignites" at a characteristic temperature T_{ign}. Then the mass of a star that barely reaches this central temperature is, from the previous equation,

$$M_{\min} \approx 2\left(\frac{3}{\pi}\right)^{1/2} \left(\frac{K_{NR}}{G^2 m_H^{8/3}}\right)^{3/4} \left(kT_{\mathrm{ign}}\right)^{3/4}$$

The ignition temperature depends on the environment of the contracting material. When the nuclear power produced in a particular region slightly exceeds the power that is lost, the region heats up and burning takes hold. If we suppose that the ignition temperature of hydrogen is about 1.5 × 10^6 K (one tenth the central temperature of the Sun), the expression above yields $M_{\min} \approx 0.02 M_{\odot}$.

Divertissement

Taking the ignition temperature to be $0.1 T_{c\odot}$ is unsatisfyingly arbitrary. Could we instead relate the ignition temperature to some intrinsic characteristic of the nuclear reaction?

You showed as a homework problem that the fusion rate may be expressed approximately as

$$r_{AB} = 6.5 \times 10^{-24} \frac{n_A n_B}{\mu_A Z_A Z_B} S(E_0) \left(\frac{E_G}{4kT} \right)^{2/3} \exp\left[-3 \left(\frac{E_G}{4kT} \right)^{1/3} \right] \text{ m}^{-3} \text{ s}^{-1}$$

If we ignore the temperature dependence of $S(E_0)$, this has the form

$$r_{AB} \propto x^{2/3} e^{-3x^{1/3}} \quad \text{where} \quad x \equiv E_G / 4kT$$

Differentiating this expression yields a maximum at $x = 0.296$ or $kT = 0.84\ E_G$.

The first step of the proton-proton chain, ${}^1\text{H}(p, e^+\nu)\,{}^2\text{D}$, has Gamow energy E_G = 0.493 MeV. Thus, the p-p chain generates maximum energy at a temperature 0.84(0.493) MeV or 4.9 × 10^9 K (!), whereas the actual burning temperature at the center of the Sun is 300 times cooler. Gravitational contraction stops when the central temperature is high enough for nuclear energy generation to supply the needed luminosity, not when the nuclear efficiency reaches some particular fraction of its maximum.

Maximum mass of a main sequence star

In Lecture 5 we derived two forms of the virial theorem for a self-gravitating system with internal pressure:

$$2K + U = 0 \quad \text{(nonrelativistic)}$$
$$K + U = 0 \quad \text{(ultrarelativistic)}$$

where K is the kinetic energy of the system, U is the potential energy, and the total energy is always $E = K + U$. [Recall that the difference between the two forms arises because pressure is equal to 2/3 of energy density for an ideal gas but only 1/3 for a completely relativistic gas such as radiation.] The total energy is negative for a gravitationally bound system.

Thus, if the internal pressure of a gravitating system is dominated by radiation pressure, the system will be on the margin between bound and unbound. Note also that contraction, which causes a nonrelativistic system to become more bound, does *not* help in this case: the total energy stays near zero. We expect such a system to be highly prone to disruption, particularly in a region of dynamic star formation where there are a variety of potentially significant external perturbations.

As in Part 2/Problem 2 of the mid-term exam, characterize the relative importance of ideal gas pressure and radiation pressure by the parameter β:

$$P = P_g + P_r \qquad P_g = \beta P \qquad P_r = (1-\beta)P$$

where $P_g = \rho kT / \mu m_H$ and $P_r = \frac{1}{3} a T^4$. From these equations it is easy to derive

$$T = \beta P \frac{\mu m_H}{\rho k} \qquad \text{and} \qquad T = \left[(1-\beta) P \frac{3}{a} \right]^{1/4}$$

and, by equating these two expressions for T,

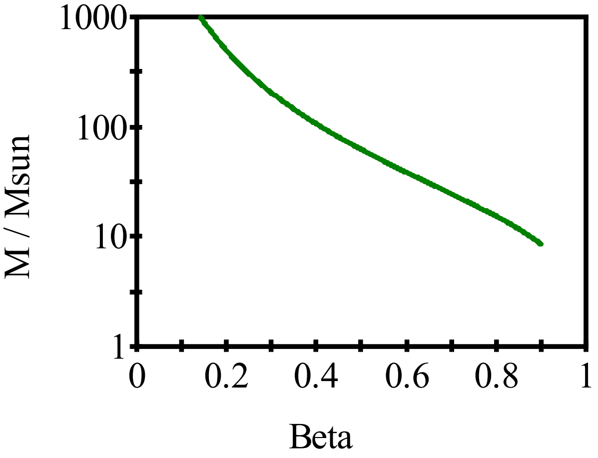
$$P = \left[\frac{3}{a} \frac{(1-\beta)}{\beta^4} \right]^{1/3} \left(\frac{k\rho}{\mu m_H} \right)^{4/3}$$

This is a general expression, valid for any combination of radiation and ideal gas pressure (except $\beta = 0$ or 1). If we equate this expression at the center of the star to the same estimate of the central pressure we used to estimate the minimum mass above, we get

$$\left[\frac{3}{a} \frac{(1-\beta)}{\beta^4} \right]^{1/3} \left(\frac{k\rho_c}{\mu m_H} \right)^{4/3} = \left(\frac{\pi}{6} \right)^{1/3} G \rho_c^{4/3} M^{2/3}$$

or

$$M = G^{-3/2}\left(\frac{k}{\mu m_H}\right)^2\left[\frac{18}{\pi a}\frac{(1-\beta)}{\beta^4}\right]^{1/2} \approx 22\left[\frac{(1-\beta)}{\beta^4}\right]^{1/2} M_\odot \quad (\mu = 0.5)$$



The graph of this function decreases monotonically with β, or, equivalently, increases monotonically with $1-\beta$ (fraction of pressure due to radiation). In the absence of an explicit stability analysis, we can make a rough guess that a star will be prone to instability if more than half the pressure is due to radiation, $1-\beta > 0.5$. The equation gives

$$\beta = 0.5 \quad \Leftrightarrow \quad M = 62\ M_\odot$$

From observation, stars more massive than 50 solar masses are quite rare.

A fundamental unit for stellar masses

We have estimated that the mass range of main sequence stars spans something less than four orders of magnitude (0.01–100 $M_\odot$), with one solar mass squarely in the middle of the range. This prompts us to ask whether there is a dimensionless number that sets the mass scale.

Recall that the fine structure constant characterizes the strength of the electromagnetic force by comparing the electrostatic potential energy between fundamental units of charge, separated by a fundamental distance, with a fundamental energy, the rest-mass energy:

$U_E = \dfrac{e^2}{4\pi\epsilon_0 r}$ electrostatic potential energy

$r = \hbar/mc$ reduced Compton wavelength

$\dfrac{U_E}{mc^2} = \dfrac{e^2}{4\pi\epsilon_0\hbar c} = \alpha$ fine structure constant

Let's apply this logic to the gravitational interaction between nucleons:

$U_G = -\dfrac{Gm_p^2}{r}$ gravitational potential energy

$r = \hbar / m_p c$ reduced Compton wavelength

$\dfrac{U_G}{mc^2} = \dfrac{Gm_p^2}{\hbar c} \equiv \alpha_G$ dimensionless constant = 5.9×10^{-39}

Another way to write α_G is $\alpha_G = \left(m_p / m_{\mathrm{Pl}}\right)^2$ where $m_{\mathrm{Pl}} = \left(\hbar c / G\right)^{1/2}$ is a fundamental length known as the Planck length.

Our expression for the minimum stellar mass can be written in terms of α_G as

$$M_{\min} \approx \frac{6\pi^{1/2}}{5^{3/4}} \left(\frac{kT_{\mathrm{ign}}}{m_e c^2}\right)^{3/4} \alpha_G^{-3/2} m_p = 0.006\, \alpha_G^{-3/2} m_p$$

if we again use the estimate $T_{\mathrm{ign}} = 1.5 \times 10^6$ K. Similar, our expression for the maximum mass becomes

$$M_{\max} = \left[\frac{270(1-\beta)}{\pi^3 \mu^4 \beta^4}\right]^{1/2} \alpha_G^{-3/2} m_p = 33\ \alpha_G^{-3/2} m_p$$

for $\beta = 0.5$ and $\mu = 0.5$.

The commonality of the expressions for $M_{\min}$ and $M_{\max}$ prompts us to introduce

$$M_* = \alpha_G^{-3/2} m_p = 1.85\, M_\odot$$

as a fundamental unit for stellar masses. Note that the number of nucleons in a fundamental stellar mass is determined solely by α_G:

$$N_* = \alpha_G^{-3/2} = 2 \times 10^{57}$$