ASTR 498N Lecture 17 As Simple As Can Be

(online at www.astro.umd.edu/~drabin/)

> *Everything should be made as simple as possible, but not simpler.*
> A. Einstein

Simplified stellar models

Older texts devote considerable space to simplified models of stellar structure that do not require knowledge of how energy is generated and transported. Before the computer era, such models were of practical importance. Nowadays, some practitioners take the point of view that simplified models are obsolete if not misleading. I think that they are still useful for developing physical understanding but should be chosen for didactic simplicity: some of the classical models (such as polytropes) require a bit of mathematical machinery to be developed and, at the end of the day, don't have analytic solutions for all values of their parameters.

We'll compromise by listing the common simplified models and then working through one of them and using the results to estimate the maximum mass of a stellar isothermal core.

Recall that the stellar structure equations divide naturally into two pairs: the mass conservation and hydrostatic equilibrium equations,

$$\frac{dM}{dr} = 4\pi r^2 \rho \qquad\qquad \frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

and the energy generation and transport equations,

$$\frac{dL}{dr} = 4\pi r^2 \rho\,\varepsilon \qquad\qquad \frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4acT^3}\frac{L}{4\pi r^2} \quad \text{or} \quad \frac{dT}{dr} = -\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr}$$

Consider the first pair equations. Rewrite the hydrostatic equilibrium equation as

$$M = -\frac{r^2}{G\rho}\frac{dP}{dr},$$

take the derivative with respect to r, and substitute for dM/dr from the mass conservation equation to yield the second-order equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho$$

From this it is evident that the pressure-density structure of a self-gravitating gas sphere can be calculated if we specify $P(r)$, or $\rho(r)$, or $P(\rho)$. Examples include:

1. *Polytrope*: $P = K\rho^{1+\frac{1}{n}}$ where K is a constant that depends on the physical nature of the polytrope, and the index n is not restricted to integer values. The resulting second-order equation is called the Lane-Emden equation. It has known analytic solutions for $n = 0$, 1 and 5; for other values of n, the equation is integrated numerically. Polytropes become more centrally concentrated as the index increases—the ratio of the central density to the mean density for $n = 0, 1, 2, 3, 5$ is 1.0, 3.29, 11.4, 54.2, ∞. The historical importance of polytropes derives from the fact that several idealized but physically relevant stellar interiors can be described by a polytropic equation of state.
    a. *Adiabatic convective equilibrium*. Recall from Lecture 12 and the mid-term exam that this state is characterized by $P = K\rho^{\Gamma_1}$. For $\Gamma_1 = 5/3$ (monatomic ideal gas), this is a polytrope of index $n = 3/2$.
    b. *Constant mixture of ideal gas and radiation pressure*. If both gas pressure and radiation pressure are significant, *and* their ratio does not vary through the star, we showed in the last lecture that the equation of state is polytropic with index $n = 3$.
    c. *Fully degenerate electron gas*. From Lecture 7, $P = K_{\mathrm{NR}}\rho^{5/3}$ or $P = K_{\mathrm{R}}\rho^{4/3}$ in the nonrelativistic or relativistic limit, respectively. These correspond to polytropes of index $n = 3/2$ or $n = 3$.
    d. *Isothermal gas sphere*. $P = \rho kT/\mu m_H = K\rho$ for T and μ constant throughout, corresponding to polytropic index $n = \infty$. This configuration (like all polytropes with $n \geq 5$) is infinite in extent and not very suitable for describing a star with a definite radius. However, an isothermal sphere is

often a fairly good approximation to the radial distribution of stars or galaxies within clusters of same.

2. *Specified run of density with radius*. The linear model considered below is an example. Other possibilities are discussed by Clayton (1986, *Am. J. Phys*. **54**, 354).
3. *Specified run of pressure with radius.* Clayton (op. cit.) shows that parameterizing the pressure *gradient* is a physically appealing way to specify the run of pressure. One choice that works well for many stars is the quasi-Gaussian form $dP/dr = -(4\pi/3)G\rho_c^2\, r\, e^{-r^2/a^2}$.

It must be emphasized that none of these simplified schemes is guaranteed to yield temperature or luminosity profiles that are consistent with a physical source of internal energy. Indeed, it is easy to produce behavior that is manifestly *in*consistent, such as a temperature profile that approaches zero at both the center and the surface of the star.

The linear model

We present the straightforward steps largely without comment. *M* and *R* are the total stellar mass and radius; *m*(*r*) is the mass interior to *r*.

$$\rho(r) = \rho_c\left(1 - r/R\right) \qquad \text{specifies model}$$

$$dm/dr = 4\pi\rho_c r^2\left(1 - r/R\right)$$

$$m(r) = \pi\rho_c r^3\left(4/3 - r/R\right)$$

$$M = m(R) = \pi R^3\rho_c/3 \qquad\qquad \rho_c = 3M/\pi R^3$$

$$m/M = 4\left(r/R\right)^3 - 3\left(r/R\right)^4$$

$$dP/dr = -Gm\rho/r^2 = -\pi G\rho_c^2 R\left[\frac{4}{3}\left(\frac{r}{R}\right) - \frac{7}{3}\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^3\right]$$

$$P(r) = P_c - \pi G \rho_c^2 R^2 \left[\frac{2}{3}\left(\frac{r}{R}\right)^2 - \frac{7}{9}\left(\frac{r}{R}\right)^3 + \frac{1}{4}\left(\frac{r}{R}\right)^4 \right]$$

$$P(R) = 0 \quad \Rightarrow \quad P_c = \frac{5}{36}\pi G \rho_c^2 R^2 = \frac{5}{4\pi}\frac{GM^2}{R^4}$$

$$P(r) = \frac{5\pi}{36} G \rho_c^2 R^2 \left[1 - \frac{24}{5}\left(\frac{r}{R}\right)^2 + \frac{28}{5}\left(\frac{r}{R}\right)^3 - \frac{9}{5}\left(\frac{r}{R}\right)^4 \right]$$

$$\begin{aligned} T(r) &= \mu m_H P(r)/k\rho(r) \\ &= \frac{5\pi}{36}\frac{G\mu m_H}{k}\rho_c R^2 \left[1 + \left(\frac{r}{R}\right) - \frac{19}{5}\left(\frac{r}{R}\right)^2 + \frac{9}{5}\left(\frac{r}{R}\right)^3 \right] \end{aligned}$$

[as may be verified from the product $\rho(r)T(r)$]

$$T_c = \frac{5}{12}\frac{\mu m_H}{k}\frac{GM}{R}$$

Gravitational potential of all gas inside radius r:

$$\begin{aligned} U(r) &= -\int_0^m \frac{Gm'}{r'} dm' = -GM \int_0^r \frac{1}{r'} \left[4\left(\frac{r'}{R}\right)^3 - 3\left(\frac{r'}{R}\right)^4 \right] 4\pi \rho_c r'^2 \left(1 - \frac{r'}{R}\right) dr' \\ &= -4\pi GM \rho_c R^2 \left[\frac{4}{5}\left(\frac{r}{R}\right)^5 - \frac{7}{6}\left(\frac{r}{R}\right)^6 + \frac{3}{7}\left(\frac{r}{R}\right)^7 \right] \end{aligned}$$

$$U(R) = -\frac{26}{35}\frac{GM^2}{R} \qquad \frac{U(r)}{U(R)} = \frac{210}{13}\left[\frac{4}{5}\left(\frac{r}{R}\right)^5 - \frac{7}{6}\left(\frac{r}{R}\right)^6 + \frac{3}{7}\left(\frac{r}{R}\right)^7 \right]$$

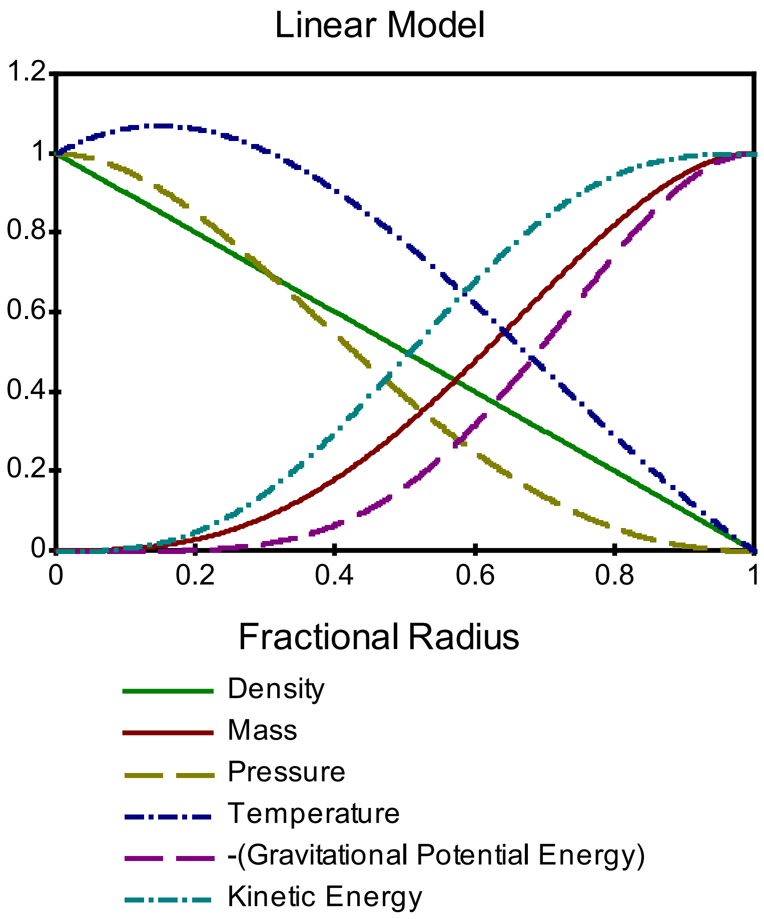
For a nonrelativistic gas, the kinetic energy density if 3/2 the pressure. Hence, the kinetic energy of all gas inside radius r is $K(r) = \frac{3}{2}\int_0^r P\,dV$ or

$$K(r) = \frac{5\pi}{24} G\rho_c^2 R^2 \int_0^r \left[1 - \frac{24}{5}\left(\frac{r'}{R}\right)^2 + \frac{28}{5}\left(\frac{r'}{R}\right)^3 - \frac{9}{5}\left(\frac{r'}{R}\right)^4 \right] 4\pi r'^2 dr'$$

$$= \frac{5\pi^2}{6} G\rho_c^2 R^2 \left[\frac{1}{3}\left(\frac{r}{R}\right)^3 + \frac{24}{25}\left(\frac{r}{R}\right)^5 + \frac{14}{15}\left(\frac{r}{R}\right)^6 - \frac{9}{35}\left(\frac{r}{R}\right)^7 \right]$$

$$K(R) = \frac{13}{35}\frac{GM^2}{R}$$

$$\frac{K(r)}{K(R)} = \frac{525}{26}\frac{GM^2}{R}\left[\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{24}{25}\left(\frac{r}{R}\right)^5 + \frac{14}{15}\left(\frac{r}{R}\right)^6 - \frac{9}{35}\left(\frac{r}{R}\right)^7 \right]$$



The runs of density, mass, pressure, temperature, and gravitational potential energy are shown at left. The model is not particularly realistic. For our present purpose it has two advantages: the physical quantities are known analytically at all radii, and the inner 30% of the mass (“core”) is roughly isothermal.

Maximum mass of a nondegenerate isothermal core

If there is no nuclear energy generation in the core of a static star—for example, at the end of the main sequence phase when the hydrogen in the core is exhausted—the core must be isothermal. This follows from two of the equations of stellar structure:

Thermal equilibrium: $dL/dm = \varepsilon \quad \Rightarrow \quad L(m) = 0$ if $\varepsilon = 0$ throughout the core

Radiative transport**:** $dT/dm \propto L \quad \Rightarrow \quad T = \text{constant}$ if $L(m) = 0$

Anywhere within a static star, the pressure at each radius must balance the pressure exerted by all the overlying mass. We now show that there is a limit to how much mass an isothermal core can support. Our main tool will be the virial theorem.

Recall (Lecture 5) that we derived the virial theorem by integrating the Lagrangian form of the hydrostatic equilibrium equation, $4\pi r^3 dP = -\left(Gm/r\right)dm$, over the whole star:

$$3\int_{P_c}^{P_s} V_r\, dP = -\int_0^M \left(Gm/r\right) dm$$

where $V_r = 4\pi r^3/3$. The right-hand integral is the gravitational potential energy of the star. We integrated the left-hand side by parts:

$$3\int_{P_c}^{P_s} V_r\, dP = 3\left[PV_r\right]_0^s - 3\int_0^{V_s} P\, dV_r$$

The second term on the right is $-2K$ for a nonrelativistic gas. We discarded the first term on the right because it vanishes strictly in the center ($V_c = 0$) and to a good approximation at the surface ($P_s << P_c$). This led to the familiar form of the nonrelativistic virial theorem, $2K + U = 0$. [Note that the linear model satisfies this equation at $r = R$.] Now, however, we take the upper limit of integration to be, not the surface, but an intermediate internal radius r_i where the pressure is not negligible. We therefore derive a more general form of the virial theorem,

$$2K_i + U_i = 3P_i V_i$$

Now consider an isothermal core of mass $M_{\rm ic}$, radius $R_{\rm ic}$, and so on. The kinetic energy of the core is simply

$$K_{\rm ic} = \frac{3}{2}\frac{M_{\rm ic} kT_{\rm ic}}{\mu_{\rm ic} m_H}$$

The gravitational potential energy may be estimated by taking the density to be constant, which leads to

$$U_{\rm ic} = -\frac{3}{5}\frac{GM_{\rm ic}^2}{R_{\rm ic}}$$

Substituting $K_{\rm ic}$ and $U_{\rm ic}$ into the surface-term version of the virial theorem and solving for $P_{\rm ic}$, we find

$$P_{ic} = \frac{3}{4\pi R_{\rm ic}^3}\left(\frac{M_{\rm ic} kT_{\rm ic}}{\mu_{\rm ic} m_H} - \frac{1}{5}\frac{GM_{\rm ic}^2}{R_{\rm ic}}\right)$$

The salient feature of this expression is that it has a maximum for a core mass

$$M_{\rm ic} = \frac{5}{2}\frac{kT_{\rm ic}}{G\mu_{\rm ic} m_H} R_{ic}$$

The maximum pressure at the boundary of the isothermal core,

$$P_{\rm ic,max} = \frac{375}{64\pi}\frac{1}{G^3 M_{\rm ic}^2}\left(\frac{kT_{\rm ic}}{\mu_{\rm ic} m_H}\right)^4 \approx 1.9\frac{1}{G^3 M_{\rm ic}^2}\left(\frac{kT_{\rm ic}}{\mu_{\rm ic} m_H}\right)^4$$

decreases as the core mass increases.

Using the linear model, we estimate the pressure of the overlying envelope at $r = 1/3$ (taken to be the edge of the core) as

$$P_{env} = 0.28\frac{GM^2}{R^4}$$

If the core is to support the envelope, we must have

$$0.28\frac{GM^2}{R^4} < 1.9\frac{1}{G^3 M_{\mathrm{ic}}^2}\left(\frac{kT_{\mathrm{ic}}}{\mu_{\mathrm{ic}} m_H}\right)^4$$

Using the linear model to write

$$T_{\mathrm{ic}} = \frac{5}{12}\frac{\mu_e m_H}{k}\frac{GM}{R}$$

and substituting in the previous equation, we find

$$\frac{M_{\mathrm{ic}}}{M} < 0.45\left(\frac{\mu_e}{\mu_{\mathrm{ic}}}\right)$$

The right-hand expression is the *Schönberg-Chandrasekhar limiting mass* (a more accurate calculation yields a coefficient of 0.38 instead of 0.45). When the core mass exceeds this limit, the pressure exerted by a nondegenerate isothermal core is insufficient to support the star, and gravitational contraction must ensue.