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**Radiative Transfer**  
(O&C §9.3)

Linear absorption coefficient (opacity)  $k_\nu$        $dI_\nu = -k_\nu I_\nu ds$        $[k_\nu] = \text{m}^{-1}$

➡ Note that  $[k_\nu^{-1}] = \text{m}$  and is a measure of the *mean free path* for photon absorption.

➡ The opacity  $k_\nu$  includes any process that removes energy from the frequency-specific pencil of radiation described by  $I_\nu$ , whether by change in direction (scattering), change in frequency, or conversion to another form of energy.

Optical depth  $\tau_\nu$        $d\tau_\nu = k_\nu ds$        $[\tau_\nu] = \text{dimensionless}$

Mass absorption coefficient  $\kappa_\nu$        $k_\nu = \rho \kappa_\nu$        $[\kappa_\nu] = \text{m}^2 \text{ kg}^{-1}$

Cross section  $\sigma_\nu$        $k_\nu = n \sigma_\nu$        $[\sigma_\nu] = \text{m}^2$   
(O&C §10.5)

Emission coefficient       $dI_\nu = j_\nu ds$        $[j_\nu] = \text{W m}^{-3} \text{ Hz}^{-1} \text{ sr}^{-1}$



$[j_\nu] \neq [k_\nu]$

**P** Does  $k_\nu d\nu = k_\lambda d\lambda$  ?

Equation of radiative transfer

$$\frac{dI_\nu}{ds} = -k_\nu I_\nu + j_\nu$$

or

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

where the *source function*  $S_\nu$  is defined as

$$S_\nu = \frac{j_\nu}{k_\nu} \quad [S_\nu] = [I_\nu]$$

➡ The radiative transfer equation is fundamentally probabilistic.

Formal solution of the radiative transfer equation:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t)e^{-(\tau_\nu-t)} dt$$

The solution is formal because, in general, we don't know  $S_\nu$ —it is part of the solution. However, we can see why “source function” is a sensible name; and, in some cases, we can make a reasonable assumption about the form of  $S_\nu$ .

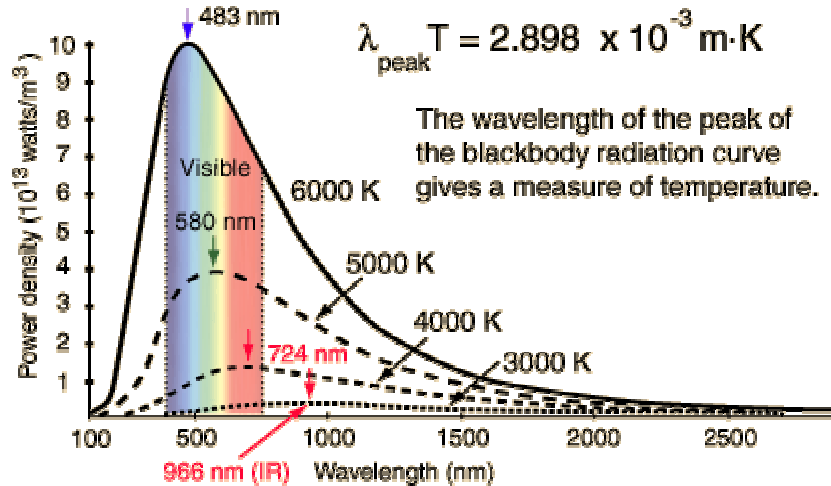
**P** O&C Problem 9.22

### Black body radiation (O&C §3.4)

Intensity  $B_\nu = \frac{c}{4\pi} u_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$  (Planck function)

Wavelength of maximum  $B_\lambda$   $T\lambda_{\max} = 2.898\text{e-}3 \text{ m K}$  (Wien displacement law)

Frequency of maximum  $B_\nu$   $Tc/\nu_{\max} = 3.670\text{e-}3 \text{ m K}$



Wien approximation ( $h\nu / kT \gg 1$ )

$$B_\nu \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

Rayleigh-Jeans approximation ( $h\nu / kT \ll 1$ )

$$B_\nu \approx 2kT / \lambda^2 \quad (\text{Note: still } B_\nu)$$

Flux density  $F = \sigma T^4 = (5.67e-8)T^4 \text{ W m}^{-2}$  (Stefan-Boltzmann law and constant)

Stellar effective temperature  $T_e$   $L = 4\pi R^2 \sigma T_e^4$

Source function for a gas in TE  $S_\nu = B_\nu$  **P** Why?

**P** Why can a star never be strictly in thermodynamic equilibrium?

**Maxwell speed distribution**  
(O&C §8.1)

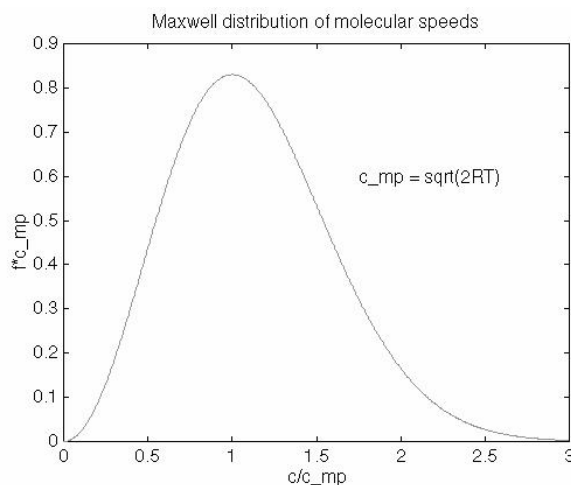
The density function of particle speed in a nondegenerate, nonrelativistic gas in TE is

$$f(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT}$$

where  $f(v)dv$  gives the probability of finding a particle of mass  $m$  in the velocity range  $v$  to  $v + dv$  at temperature  $T$ . Since it's a probability,  $\int_0^{\infty} f(v)dv = 1$ .

**P** What is  $v_{mp}$ , the most probable speed for a particle of mass  $m$  and temperature  $T$ ? Write this also as a convenient numerical formula for electrons and for hydrogen atoms.

If we normalize all speeds to the most probable speed,  $x = v/v_{mp}$ , the density of  $x$  has a universal shape:



The Maxwell speed density is derived from the more fundamental 6-dimensional phase space density describing the equilibrium of an ideal gas,

$$f(\mathbf{v}) d^3\mathbf{v} d^3\mathbf{x} = \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} d^3\mathbf{v} d^3\mathbf{x} \quad (\text{Maxwell-Boltzmann})$$

where  $v^2 = v_x^2 + v_y^2 + v_z^2$ .  $f(\mathbf{v})$  does not depend on the spatial coordinates  $\mathbf{x}$  because there is no spatial (or time) dependence in a true equilibrium. Note that this density is a pure Gaussian.

**P** Derive the density function for the radial (line-of-sight) velocity (which governs spectroscopic Doppler shifts) and for the kinetic energy  $E = \frac{1}{2}mv^2$ .

The Maxwell-Boltzmann density function is approximately applicable in a wide variety of circumstances. However, we will find that the underlying *Maxwell-Boltzmann*

*statistics* are not adequate to describe the state of all stellar interiors. We will also need *Fermi-Dirac* statistics and *Einstein-Bose* statistics (the Planck radiation function, above, can be derived from Einstein-Bose statistics, which apply to a photon gas).

### **Boltzmann formula (excitation)**

(O&C §8.1)

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\Delta E/kT}$$

where  $u$  (upper) and  $l$  (lower) label two energy levels of an atom or ion,  $n$  ( $\text{m}^{-3}$ ) is the population of the level,  $g$  is the statistical weight of the level (the number of states with different quantum numbers but the same energy), and  $\Delta E = E_u - E_l$  is the energy difference between the levels.

The Boltzmann formula holds for thermodynamic equilibrium at temperature  $T$ . Sometimes an *excitation temperature*  $T_{ex}$  is defined through this relationship even when TE is not known to hold, but such a temperature applies only to the specific levels involved.

### **Saha equation**

(O&C §8.1)

$$\frac{n_e n_{r+1}}{n_r} = \frac{2u_{r+1}}{u_r} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT} \quad (\text{Saha})$$

describing the relative number densities of two ionization states of the same ion,  $r$ -times ionized and  $(r+1)$ -times ionized, in terms of the ionization potential  $\chi_r$ , the temperature  $T$ , and the *partition functions*  $u_{r+1}$  and  $u_r$ , where

$$u_r(T) = \sum_n g_{r,n} e^{-E_n/kT}$$

The partition function is a summation of the statistical weights of all the bound states of the ion, each weighted by a Boltzmann factor to account for the relative population the level relative to the ground state. Some authors (including O&C) use  $Z$  instead of  $u$  to denote a partition function.



The Saha equation is sometimes multiplied through by  $kT$  and written in terms of  $P_e$ , the electron pressure. This assumes the perfect gas law,  $P_e = n_e kT$ , and is not correct if, for example, the gas is partially degenerate.

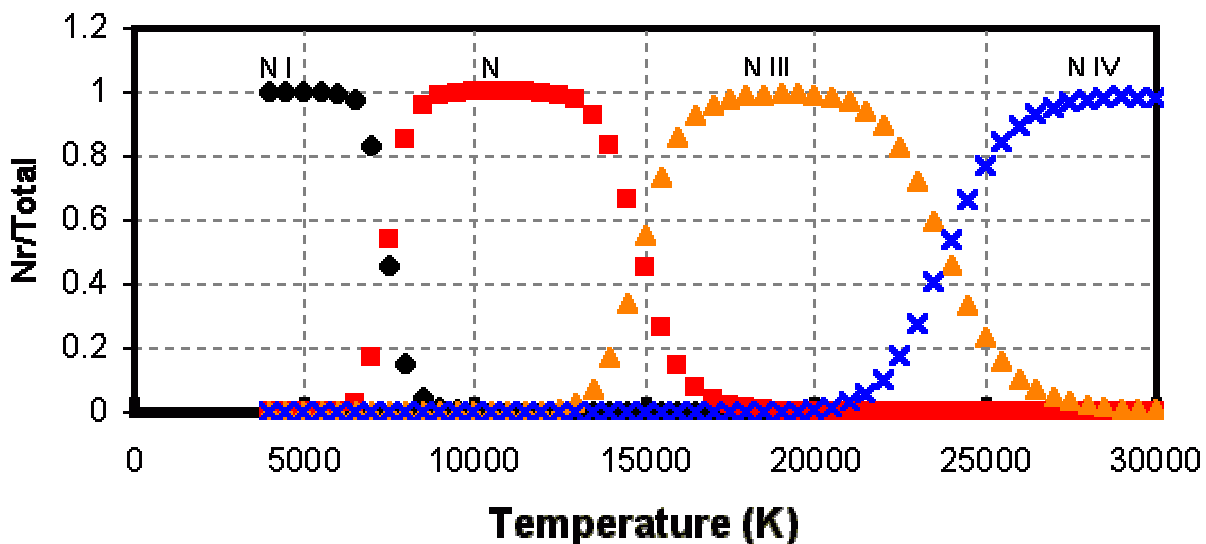
For numerical work, it's convenient to have the Saha equation in numerical form, e.g.,

$$\log\left(\frac{n_e n_{r+1}}{n_r}\right) = 21.68 + \log\left(\frac{u_{r+1}}{u_r}\right) + 1.5 \log T - \frac{5040}{T} \chi_r$$

where  $n$  is in  $\text{m}^{-3}$  and  $\chi_r$  is in eV. This is the origin of the odd factor  $5040/T$  that crops up in astronomical papers. Notation: “log” means  $\log_{10}$ , “ln” means  $\log_e$ .

It's a useful fact, illustrated for atomic nitrogen below (the label above the red points should be N II, not N), that one rarely needs to consider more than two stages of ionization at once in a stellar interior or atmosphere.

### Nitrogen ( $P_e=1 \text{ dyne cm}^{-2}$ )



The Saha equation applies to thermodynamic equilibrium. As in the case of the Boltzmann formula, an *ionization temperature*  $T_i$  can be defined with reference to a specific ion and levels.

**P** Derive an expression for  $n_{r,k}/n_r$ , the population of an individual level  $k$  of an  $r$ -times ionized species relative to the total population of that ion.

**P** O&C Problems 8.10(a), 8.10(b), and 8.(12).