ASTR 498N   Lecture 5   So What's a Virial?

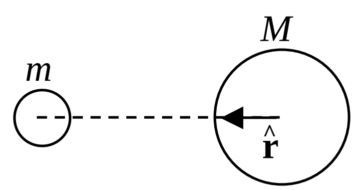
(online at www.astro.umd.edu/~drabin/)

Gravity is nothing if not persistent.
Charles Bogle

We're sneaking up on stellar structure. Much of a star's life (like ours, as you will eventually discover) consists of a valiant struggle to resist the tireless force of gravity.

Free fall
(O&C §2.2)

What Newton says, goes: $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$



As you know, the gravitational field at a point outside (or, in fact, within) a spherically symmetric shell or ball of mass is the same as if all the mass interior to that point were concentrated at the center. Consider the free-fall of the mass *m* from rest at initial radius R_0 (we'll assume $M >> m$ for simplicity, although everything goes through the same in the center-of-mass frame).

Acceleration of *m*: $m\frac{d^2r}{dt^2} = -\frac{GMm}{r^2}$

We already know a first integral, $\frac{1}{2}mv^2 - \frac{GmM}{r} = K + U = \text{constant} = \frac{GmM}{R_0}$

where *K* is the kinetic energy and *U* the gravitational potential energy of the system,

from which we can write $\frac{dr}{dt} = -\left[2GM\left(\frac{1}{r} - \frac{1}{R_0}\right)\right]^{-1/2}$

and immediately integrate:

$$t_{\mathrm{ff}} = \int_{r_0}^{0} \frac{dt}{dr}\,dr = -(2GM)^{-1/2} R_0^{3/2} \int_{r_0}^{0} \left(\frac{1}{r/R_0} - 1 \right)^{-1/2} d\left(r/R_0\right)$$

$$= \left(R_0^3/2GM\right)^{1/2} \int_0^1 \left(\frac{x}{1-x} \right)^{1/2} dx \qquad \text{where } x = r/R_0$$

$$= \left(\frac{R_0^3}{GM} \right)^{1/2} \left(\frac{\pi}{2^{3/2}} \right) \qquad (\text{let } x = \sin^2\theta \text{ above})$$

We can write this in terms of the mean mass density inside R_0, $\bar{\rho} = M\big/\left(\frac{4}{3}\pi R_0^3\right)$, as

$$t_{\mathrm{ff}} = (G\bar{\rho})^{-1/2} (3\pi/32)^{-1/2}$$

Apart from the numerical factor, *this is an equation worth remembering*! "One over root G rho" pops up again and again as a characteristic dynamical time in a gravitational system; for example, it applies to the period of a radially pulsating star. Such a general relationship must be derivable from order-of-magnitude reasoning.

R O M (Rough Order of Magnitude estimate)

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \qquad \frac{d^2r}{dt^2} \approx \frac{-R}{t^2} \qquad \Rightarrow \qquad t_{\mathrm{ff}} \approx \left(R^3/GM\right)^{1/2}$$

If you try to construct a quantity with the dimensions of time from G, M, and R (the only primary quantities available), you'll get the same estimate.

Hydrostatic equilibrium
(O&C §10.1)

What can resist gravity? Pressure! Consider a shell of mass within a spherical star. The shell is at radius r with thickness dr, local mass density $\rho(r)$, and mass M_r interior to r. The downward gravitational force on the shell is

$$-\frac{GM_r}{r^2} 4\pi r^2 \rho\, dr$$

while the pressure force supporting the shell is

$$4\pi r^2\, dP$$

In hydrostatic equilibrium, these forces balance:

$$\boxed{\frac{dP}{dr} = -\frac{GM_r \rho}{r^2} \qquad \text{hydrostatic equilibrium}}$$

This is a first-order differential equation and therefore requires a boundary condition, such as the central pressure P_c.

We can write this in another useful way. From

$$M_r = \int_0^r 4\pi r^2 \rho(r')\, dr'$$

we have

$$\boxed{\frac{dM_r}{dr} = 4\pi r^2 \rho \qquad \text{mass conservation}}$$

Dividing the first boxed equation by the second gives

$$\boxed{\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4 (M_r)} \qquad \text{hydrostatic equilibrium (Lagrangian)}}$$

where the notation stresses that, in this form, the interior mass M_r is the independent variable and r is dependent. This so-called Lagrangian ("follow the mass") form of the equation is often convenient.

Virial theorem

(O&C §2.4)

Start from the Lagrangian form of the hydrostatic equilibrium equation, written as

$$4\pi r^3 dP = -\left(GM_r / r\right) dM_r$$

and integrate over the whole star:

$$3\int_{P_c}^{P_s} V_r\, dP = -\int_0^{M_s} \left(GM_r/r\right) dM_r \qquad (*)$$

where V_r is defined analogously to M_r, and P_s and M_s refer to the surface of the star. The integrand on the right hand side is just the gravitational potential energy of a shell with mass dM_r at a distance r from the center of a spherical mass distribution of total mass M_r. The integral is therefore the gravitational potential energy U of the star as a whole.

Integrate the left hand side by parts:

$$3\int_{P_c}^{P_s} V_r\, dP = 3\left[PV\right]_c^s - 3\int_0^{V_s} P\, dV_r$$

The integrated part on the right vanishes strictly at the lower limit (because $V_c = 0$) and to a very good approximation at the upper limit (because $P_s << P_c$). For a nonrelativistic gas, $3P$ is equal to twice the thermal (kinetic) energy density, $2K$.

Altogether then, we can write equation (*) above as

$$\boxed{2K + U = 0 \quad \text{virial theorem (nonrelativistic)}}$$

For a relativistic (e.g., photon) gas, pressure is equal to one third (not two thirds) of the energy density, so instead we would have

$$\boxed{K + U = 0 \quad \text{virial theorem (relativistic)}}$$

Now the virial theorem (from L. *vis vires*: force, power, strength) is more general than the above derivation would suggest and applies to a wide variety of gravitating systems in equilibrium—for example, binary star orbits or a cluster of galaxies. However, the common derivation in terms of point masses (as in O&C §2.4) makes the result as it applies to stars appear unnecessarily mysterious. [For the record, the *virial of Clausius* for a system of point masses is $\sum_i \mathbf{F_i} \cdot \mathbf{r_i}$ — so there.]

Consequences of the virial theorem

For now, we'll restrict attention to nonrelativistic systems. Since the total energy is $E = K + U$, we can also write the virial theorem as $E + K = 0$.

Consider a slowly contracting star, nearly in equilibrium so that the virial theorem is satisfied at every stage. If the gravitational potential energy changes by ΔU (negative), the thermal energy must change by $\Delta K = -\frac{1}{2}\Delta U$ (positive). The star heats up! The change in the total energy is $\Delta E = \Delta K + \Delta U = \frac{1}{2}\Delta U$ (negative). The star as a whole must lose energy in some form (usually, radiation).

Thus we have the Three-Fold Way of quasistatic gravitational contraction:

1. The star gets hotter.
2. Energy is liberated from the system.
3. The total energy of the system decreases—i.e., the star becomes more tightly bound.

In thermodynamic terms, a star has negative specific heat.

Jeans mass (O&C §12.2)

Consider a spherical cloud (protostar) of constant density. If $2K < -U$, the kinetic agitation in the cloud will not suffice to overcome gravity, and the cloud will collapse. Using the dimensional estimate $U \approx -GM^2/R$ and the internal kinetic energy of a perfect gas, $K = \frac{3}{2}\frac{MkT}{\mu m_H}$, the condition for collapse is

$$\frac{3MkT}{\mu m_H} < \frac{GM^2}{R}$$

If we eliminate the mass through $M = 4\pi R^3\rho/3$, we estimate the *Jeans length*,

$$R_J \approx \left(\frac{9}{4\pi}\frac{kT}{\mu m_H}\frac{1}{G\rho}\right)^{1/2}$$

such that a cloud of density ρ and temperature T will collapse if $R > R_J$. Note our old friend "one over root G rho."

The corresponding *Jeans mass* is

$$M_J \simeq 4\pi R_J^3 \rho / 3$$

Stars by R O M

(O&C §10.3)

Central pressure

From the Lagrangian form of the hydrostatic equilibrium equation, we estimate

$$P_c \approx \frac{1}{4\pi} \frac{GM^2}{R^4}$$

This expression underestimates the central pressure of the Sun by about two orders of magnitude; it's closer to the pressure at the half-radius point (as we'll see, stars are centrally concentrated). In general, we must remember that R O M means what it says.

Central temperature

Temperature is related to pressure through an equation of state, $P = P(T,\rho,C)$ where C represents chemical composition. For pressure arising from a perfect gas and radiation (which applies in most parts of most stars),

$$P = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$$

where $a = 4\sigma/c$ and μ is the mean molecular weight. Neglect radiation and use the mean density $\bar{\rho} = M / \left(4\pi R^3/3\right)$ and our earlier estimate for P_c to derive

$$T_c \approx \frac{\mu m_H}{k} \frac{GM}{3R}$$

For a gas of pure ionized hydrogen, $\mu = ½$. For the Sun, this estimate gives about 4e6 K, a factor of four lower than the standard model.

P Repeat this calculation for pure radiation pressure. Is radiation pressure more important for high or low-mass stars?

Kelvin-Helmholtz timescale

(O&C §10.3)

The present gravitational potential energy of the Sun is, dimensionally, $U_\odot = -GM_\odot^2 / R_\odot \simeq 4\text{e}41\ \text{J}$. We know from the virial theorem that the Sun had to release half this much energy during contraction. The present luminosity of the Sun is 4e26 W. If that luminosity also applied in the past, the Sun would have radiated away the necessary amount of energy in 1e15 sec or 30 million years. This is known as the Kelvin-Helmholtz time scale.

At the time Darwin published *On the Origin of Species* in 1859, most astronomers believed that the Sun was in fact powered by gravitational contraction. Kelvin and Darwin had several public disagreements about the extended timescales required for biological evolution in Darwin's picture. The advent of radioactive dating settled the issue (for scientists), showing that the Earth is several billion years old.

In 1929, Eddington noted that, if the Sun could convert all its rest mass to energy according to $E_\odot = M_\odot c^2$, the Sun could shine at its present rate for ~1e13 y. Thus, a subatomic process that converts 10% of the mass of the Sun to energy with about 0.1% efficiency would suffice to power the Sun for a geological time scale.