(online at www.astro.umd.edu/~drabin/)

Well, isn't this is a fine state.

A *pressure equation of state* (usually shortened to equation of state) relates the pressure to the temperature, mass density, and composition of a gas in thermodynamic equilibrium: $P = P(T,\rho,C)$. Because T, ρ, and C vary within a star, and the star is radiating energy, a star cannot be in strict thermodynamic equilibrium (TE). Usually, however, it makes sense to consider the gas to be at a particular location to be very nearly in TE; this approximation is called *local thermodynamic equilibrium* (LTE).

To justify the LTE approximation with one example, we consider the variation of temperature in the solar interior. A characteristic microscopic length scale is the mean free path (mfp), $\ell = 1/n\sigma$, where σ is a cross section. If we consider a point in the Sun at about half its radius, $T \approx 10^{6.5}$ K and $n_e \approx 10^{30}$ m^{-3}. The mfp for photons is limited mainly by scattering off free electrons (Thomson scattering), for which $\sigma \approx \pi r_0^2$ where $r_0 = e^2/m_e c^2 = 10^{-14.6}$ m is the so-called classical electron radius. The resulting mfp is $\ell = 10^{-1.3}$ m. Comparing this with a characteristic temperature gradient, $T_{\rm cen} / R \approx 10^{-2.6}$ K m^{-3}, we see that the temperature is expected to vary only minutely over the range of a typical photon.

Mean molecular weight

(O&C §10.2)

The mean molecular weight μ in a gas is just the average mass of a free particle in units of m_H, the mass of a hydrogen atom. Thus, the number density of free particles is

$$n = \rho / \mu m_H$$

Let X_Z represent the fraction *by mass* of the element with atomic number Z, such that $\sum_Z X_Z = 1$. Let N_Z be the *number* of free particles contributed to the gas by each atom of element Z. If all such atoms are neutral, $N_Z = 1$. For complete ionization, $N_Z = Z + 1$. The number *density* of atoms of element Z is

$$n_Z = \rho_Z / A_Z m_H$$

where A_Z is the atomic weight.

With these definitions, the total number density of free particles is

$$n = \sum_Z N_Z n_Z = \frac{\rho}{m_H} \sum_Z \frac{X_Z N_Z}{A_Z}$$

Comparing this with $n = \rho / \mu m_H$, we have

$$\boxed{\mu^{-1} = \sum_Z \frac{X_Z N_Z}{A_Z}}$$

It is conventional in astrophysics to assign special labels to the following mass fractions:

$$X_1 \equiv X \qquad X_2 \equiv Y \qquad 1 - X - Y \equiv Z'$$

(we use Z' here instead of the usual Z to avoid confusion with the nuclear charge). With these definitions,

$$\mu^{-1} = \frac{XN_H}{1.008} + \frac{YN_{He}}{4.004} + Z' \left\langle \frac{N_Z}{A_Z} \right\rangle$$

where $\langle N_Z / A_Z \rangle$ is an average over $Z > 2$, weighted by X_Z. It is a convenient empirical fact that $A_Z \approx 2Z + 2$ (check the periodic table).

In general, $\mu = \mu(T, P, C)$ since temperature and pressure determine the degree of ionization and C determines what there is to ionize. Consider two extreme cases: a neutral gas and a completely ionized gas. The completely ionized limit is a good approximation over much of a stellar interior.

Completely neutral

$N_Z = 1$ for all Z. For the relative abundances of heavy elements in the Sun, $\langle 1/A_Z \rangle \approx 1/15.5$ and

$$\mu^{-1} = X + Y/4 + Z'/15.5$$

To the extent that the relative heavy-element abundances vary in lockstep as Z' ("metallicity") changes—true to first approximation—this expression applies to any Z'.

Completely ionized

Here $N_H = 2$, $N_{He} = 3$, $N_Z = Z + 1$. With $A_Z \approx 2Z + 2$, $\langle N_Z / A_Z \rangle = 0.5$ and

$$\boxed{\begin{aligned}\mu^{-1} &= 2X + 0.75Y + 0.5Z' \\ &= 0.5 + 1.5X + 0.25Y\end{aligned}}$$

independent of T and p. For typical Population I composition ($X = 0.70$, $Y = 0.28$, $Z' = 0.02$), $\mu \approx 0.62$. For pure H, $\mu = 0.5$. For pure He, $\mu = 1.33$. For pure heavy elements with solar relative abundances, $\mu = 2$. Thus, $0.5 < \mu < 2$ for complete ionization.

Perfect gas

(O&C §10.2)

A perfect gas is one in which there are no interactions between particles in the gas. Although this criterion is never satisfied exactly, it is physically sound if the average interaction energy between particles is much smaller than their kinetic energies. In a stellar interior, particles interact mainly via Coulomb potential energy, which is typically much smaller than kT.

There's one immediate use for the μ-nastics above—the perfect gas equation of state:

$$P_{\text{gas}} = nkT = \frac{\rho kT}{\mu m_H}$$

P_{gas} depends on the composition of the gas through μ. Note that the perfect gas equation of state holds for a classical gas even when the particles are relativistic.

P Show that $P_{\text{gas}} = 2\,u_{\text{gas}} / 3$ for non-relativistic particles and $P_{\text{gas}} = u_{\text{gas}} / 3$ for ultra-relativistic particles, where u_{gas} is the kinetic energy density.

Photon gas

(O&C §10.2)

$$P_{\rm rad} = \frac{1}{3} u_{\rm rad} = \frac{1}{3} a T^4 = \frac{1}{3} \frac{4\sigma T^4}{c}$$

where $u_{\rm rad}$ is the energy density. $P_{\rm rad}$ is a function of T only.

Equilibrium particle statistics

Recall that the Maxwellian speed distribution for a classical gas followed from the more fundamental velocity-space density, $f(\boldsymbol{v})\, d^3\boldsymbol{v}$. In quantum or relativistic systems, it is more natural to work with the density of states in momentum space, $g(\boldsymbol{p})\, d^3\boldsymbol{p}$, giving the number of states per unit spatial volume with momentum $\boldsymbol{p}$ in the range dp_x, dp_y, dp_z.

It is shown in most quantum mechanics texts that the number of distinct states of a free particle in a box of linear dimensions $dxdydz$ and in the momentum range $dp_x dp_y dp_z$ is $1/h^3$; that is, each state occupies a volume h^3 in phase space. This is a consequence of the uncertainty principle. In the terminology above, $g(\boldsymbol{p}) = 1/h^3$. However, protons, neutrons and electrons also have spin, an internal degree of freedom that doubles the number of possible states. Similarly, photons can have two distinct states of polarization. Thus, the correct expression is $g(\boldsymbol{p}) = 2/h^3$.

Because an equilibrium gas is isotropic, we may integrate over momentum shells to obtain the density function of $p = |\boldsymbol{p}|$,

$$g(p)\, dp = \frac{2}{h^3} 4\pi p^2\, dp$$

Now, this expression enumerates all possible states. Integrating $g(p)dp$ over all p gives the (infinite) volume density of *states*, not the spatial number density of *particles*. To get the spatial volume density of particles, we have to multiply $g(p)dp$ by the probability $f(p)$ that a state of momentum p will be occupied by a particle. That is,

$$n = \int_0^\infty f(p)\, g(p) dp$$

The *occupation index* $f(p)$ is derived in statistical mechanics for the two fundamental forms of quantum statistics:

$$f(p) = \frac{1}{\exp\left[\left(\varepsilon_p - \psi\right)/kT\right] - 1} \qquad \text{Bose-Einstein (identical bosons)}$$

$$f(p) = \frac{1}{\exp\left[\left(\varepsilon_p - \psi\right)/kT\right] + 1} \qquad \text{Fermi-Dirac (identical fermions)}$$

where $\varepsilon_p^2 = p^2c^2 + m^2c^4$ is the relativistic expression for energy and ψ is the chemical potential that appears in the thermodynamic relationship $dE = TdS - PdV + \psi dN$. Note that ψ has the dimensions of energy (per particle). Note also that $f(p)$ is not a density function: $f(\varepsilon_p) = f(p)$, where ε_p is the energy corresponding to p.

The occupation index for every state will be small if it is small for even the lowest energy state, $\varepsilon_p = mc^2$; that is, if $\exp\left[\left(mc^2 - \psi\right)/kT\right] \gg 1$, in which case

$$f(p) = \exp\left[-\left(\varepsilon_p - \psi\right)/kT\right] \qquad \text{Maxwell-Boltzmann (classical statistics)}$$

Thus, for both fermions and bosons, the condition for a classical gas is

$$\exp\left[\left(mc^2 - \psi\right)/kT\right] \gg 1.$$

P Use the normalization condition $n = \int_0^\infty f(p)\,g(p)\,dp$ to derive an expression for the chemical potential ψ of a classical gas of non-relativistic particles and show that the condition for a classical gas demands that the average separation of gas particles is large compared with their typical de Broglie wavelength.