(online at www.astro.umd.edu/~drabin/)

In every government on earth is some trace of human weakness, some germ of corruption and degeneracy, which cunning will discover, and wickedness insensibly open, cultivate and improve.
Thomas Jefferson

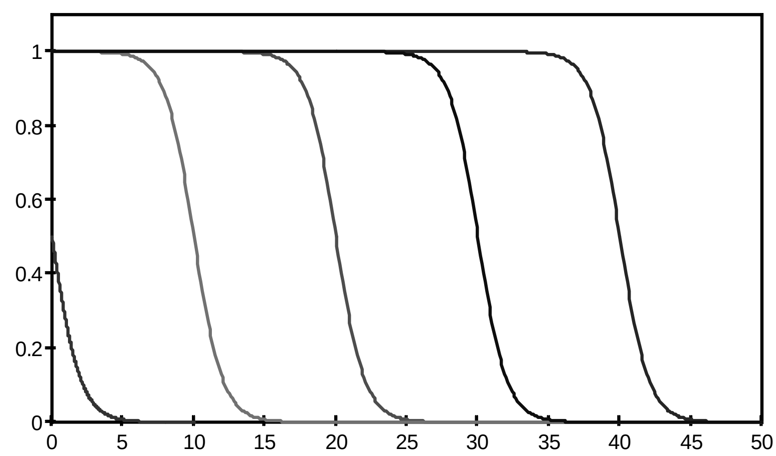
Degenerate electron gas

(O&C §15.3)

As electrons are spin-½ particles, an electron gas obeys Fermi-Dirac statistics. Note that $f(p) \leq 1$, an expression of the Pauli exclusion principle. When $f(p) = 1$, all the available states are occupied.

Consider the behavior of the Fermi-Dirac occupation index

$$f(\varepsilon_p) = \frac{1}{\exp\left[\left(\varepsilon_p - \psi\right)/kT\right] + 1}$$



(often called the Fermi function) for $\psi / kT >> 1$. The graph shows $f(\varepsilon_p)$ for ψ = 0, 10, 20, 30, 40 (left to right) where both ε_p and ψ are measured in units of kT. Always, $f(\varepsilon_p) = ½$ when $\varepsilon_p = \psi$. The function drops to nearly zero over a range of a few kT around $\varepsilon_p = \psi$, which makes the function look more and more like a step function as ψ / kT gets larger. For Fermi-Dirac statistics, ψ is called the *Fermi energy*, ε_F, and the corresponding momentum is the Fermi momentum, p_F .

The limiting case where $\psi / kT >> 1$ and $f(\varepsilon_p)$ can be treated as a step function is called *complete degeneracy*.

In complete degeneracy, $f(p)g(p)$ is a step function and so, therefore, is the electron density:

$$n_e(p)\,dp = \begin{cases} \left(2/h^3\right) 4\pi p^2 dp & p < p_F \\ 0 & p > p_F \end{cases}$$

The total electron density is

$$n_e = \int_0^{p_F} n_e(p)\,dp = \frac{8\pi}{3h^3} p_F^3$$

which can also be solved for the Fermi momentum (or energy) as a function of electron density,

$$p_F = \frac{h}{2}\left(\frac{3}{\pi}\right)^{1/3} n_e^{1/3}$$

In a degenerate electron gas, the electron density can only increase if the Fermi momentum increases. This makes sense if we think of complete degeneracy as a *low-temperature* limit (remember, the criterion is $\psi / kT >> 1$). At absolute zero, the gas is in its ground state of minimum energy. Because the exclusion principle allows only one particle per possible state, the lowest energy is obtained by piling particles into successively higher momentum (and energy) states until all the particles are accommodated. The last particle added has an energy comparable to ψ, which may be quite large. Voila! —pressure without temperature! This effect is called *degeneracy pressure.*

In this sense, a quantum gas is a cold gas—but the standard of coldness depends on density. At high enough density, a billion degrees can be "cold." To see this, anticipate the result of one of the problems given in the last lecture: a gas behaves classically if the average particle separation is large compared to a typical de Broglie wavelength, $\lambda = h / p$. For a nonrelativistic gas in thermal equilibrium, $p = mv = (mkT)^{1/2}$. Thus, $n^{-1/3} \approx h(mkT)^{-1/2}$ or $n_d \approx (mkT/h^2)^{3/2}$ marks the onset of degeneracy. When a star like the Sun leaves the main sequence, it develops a helium core with $T \approx 10^8$ K and $n_e \approx 10^{34.5}$ m^{-3}.

However, $n_d \approx 10^{33.3}$ m^{-3} at this temperature, so the electrons feel cold and degenerate despite the toasty warmth.

The pressure may be calculated from the fundamental integral (O&C Eq. 10.9),

$$P_e = \frac{1}{3}\int_0^\infty v_p\, p\, n_e(p)\, dp$$

where v_p is the speed corresponding to momentum p. There are two limiting cases according to how v_p and p are related.

Nonrelativistic complete degeneracy

If $\varepsilon_F \ll m_e c^2 = 0.51\ \text{MeV}$, $v_p = p / m_e$ and the pressure integral is easy:

$$P_{e,\text{nr}} = \frac{8\pi}{15 m_e h^3}\, p_F^5 = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3}$$

Ultrarelativistic complete degeneracy

In this case we may set $v_p = c$ and the pressure integral is again easy:

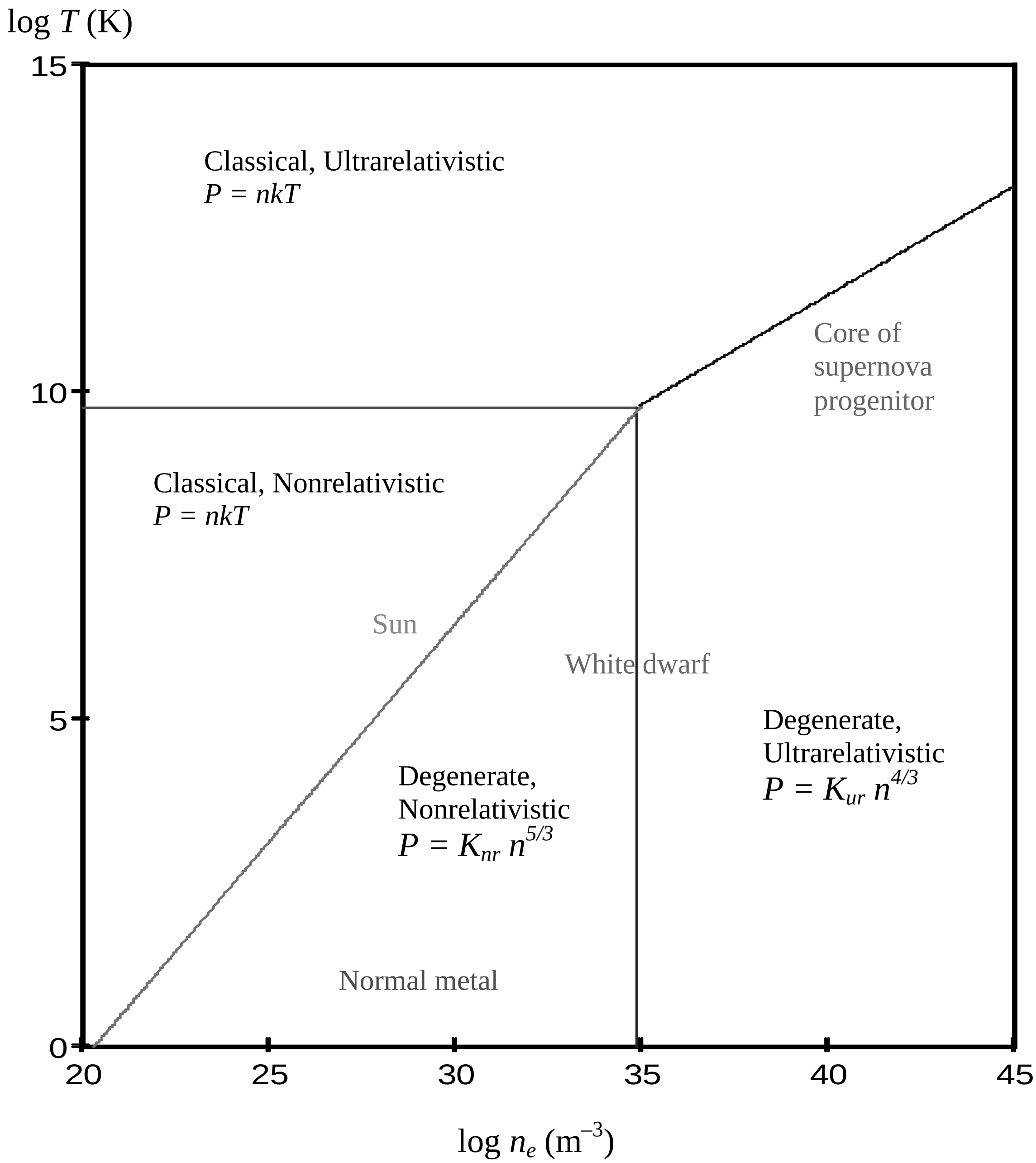
$$P_{e,\text{ur}} = \frac{2\pi}{3h^3}\, p_F^4 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}$$

The general case for any v/c may also be solved analytically, but this is more detail than we need. It is also possible to treat partial degeneracy, where the occupation index is intermediate between a step function and the classical exponential. We'll let that slide too.

Note three things about the two preceding equations:

1. They are equations of state for electron pressure in a perfect, completely degenerate electron gas.
2. They do not involve temperature.
3. The pressure rises with density faster than it does in a non-degenerate gas—the degenerate equation of state is "stiffer."

Regimes of electron degeneracy in the density-temperature plane



When considering the hydrostatic support of a star, we must also add in the pressure of the nuclei that accompany the electrons. Since nuclei are never degenerate except in the most exotic objects [why?], the nuclear pressure is classical. However, we must remember to exclude the electrons from the mean molecular weight because we have already accounted for their pressure.

Electron degeneracy has some spectacular consequences for the late stages of stellar evolution. For example, when the temperature in the core of the Sun becomes hot enough to burn helium, the electrons are degenerate and provide most of the pressure. The helium fusion increases the energy generation in the core and causes the temperature to rise. How will the star react? In a non-degenerate gas, the pressure would increase and act to expand the core, thus lowering the temperature (by the virial theorem) and acting as a thermostat. When most of the pressure is from degeneracy, however, increasing the temperature has almost no effect on the total pressure. There is no thermostatic effect. The resulting thermal runaway is called a *helium flash*.

Another surprising effect of degeneracy involves energy transport. We will find that, in ordinary circumstances, energy is transported mainly by radiation and convection. Heat conduction is very inefficient because the mean free path for particle collisions is so small. However, in a degenerate electron gas, an energetic electron has a very long free path because it is difficult for the electron to lose energy: almost all of the lower-energy states are already filled. This leads to high thermal conductivity and a strong tendency for the gas to be isothermal.