ASTR 498N Lecture 8 The way forward

(online at www.astro.umd.edu/~drabin/)

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The way to the stars is open.
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The equations of stellar evolution
(O&C §10.5)

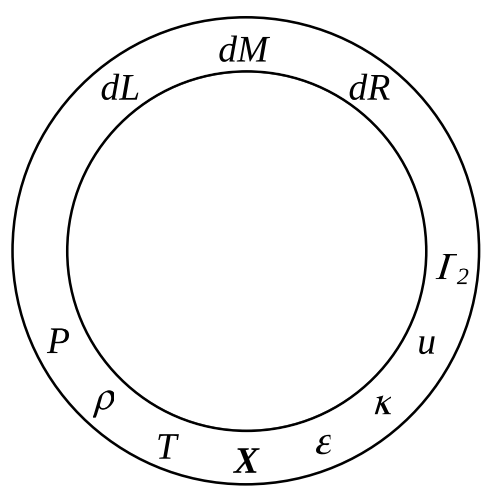
We've already derived two of the four equations of static stellar structure, expressing mass conservation and hydrostatic equilibrium. Although we still need to assemble more ingredients—concerning the generation and transport of energy—let's write down a full set of evolution equations to get an overall sense of the enterprise at hand.

We'll consider only spherically symmetric stars, in which all physical variables are constant on spherical shells of constant radius (or internal mass). We also assume the star does not rotate.

What does it mean to specify stellar structure?

We will consider the instantaneous structure of a star to be specified if the following *structural variables* are known as functions of r or M :

Pressure	$P(r)$	or	$P(M)$
Density	$\rho(r)$	or	$\rho(M)$
Temperature	$T(r)$	or	$T(M)$
Mass or radius	$M(r)$	or	$r(M)$
Luminosity	$L(r)$	or	$L(M)$
Composition	$\boldsymbol{X}(r)$	or	$\boldsymbol{X}(M)$



where $\boldsymbol{X}$ is an n-vector of element mass fractions with $\sum_{Z=1}^{n} X_Z = 1$. Since M and L are variables, we'll need to use other symbols ($\mathcal{M}$ and $\mathcal{L}$) for the total mass and luminosity of the star; for notational consistency, we'll also use $\mathcal{R}$ for the total radius. These choices are not universal. For example, some authors use m or M_r in place of M, or L_r or ℓ in place of L.

How many differential equations will be required?

There are $n + 5$ structural variables. Of the n composition variables, only n–1 are independent because they sum to unity. We'll show below that nuclear reaction rates determine n–1 independent equations of the form $dX_Z/dt = f(\rho, T, \boldsymbol{X})$, leaving five structural variables. The pressure equation of state, $P = P(\rho, T, \boldsymbol{X})$, provides one relationship these remaining variables. We therefore expect to need four differential equations in addition to the composition equations.

The differential equations

Conservation of mass

$$\boxed{\frac{dM}{dR} = 4\pi r^2 \rho}$$

Conservation of momentum

$$\boxed{\rho \frac{d^2 r}{dt^2} = -\left(\frac{dP}{dr} + \frac{GM\rho}{r^2}\right)}$$

We earlier derived this in the form where the left hand side vanishes: hydrostatic equilibrium. If the pressure gradient does not exactly balance the gravitational force on the shell, the shell accelerates (Newton's law). Note that the time derivative of $r = r(M)$ implies an Lagrangian rather than Eulerian description.

Conservation of energy

Write the first law of thermodynamics as $\delta E = \delta Q - \delta W$, where δ is an infinitesimal Eulerian change in a shell of mass dM, luminosity dL, and volume dV. Substitute for the thermodynamic quantities to obtain

$$\delta(u\, dM) = (\varepsilon\, \delta t\, dM - dL\, \delta t) - P\delta(dV)$$

Using $dM = \rho\, dV$ and $\delta(dM) = 0$, we can rewrite this as

$$\delta u\, dM = \left(\varepsilon - dL/dM\right)\delta t\, dM - P\delta(1/\rho)\, dM$$

So, for an infinitesimal change over δt,

$$\boxed{\frac{du}{dt} - P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{dL}{dM}}$$

For the static case ($d/dt = 0$), this just says that the luminosity dL added by shell dM is due to the energy generation rate per unit mass. From the thermodynamic relation $TdS = dE + pdV$ for quasistatic changes, the equation can also be expressed as

$$\boxed{\frac{dL}{dM} = \varepsilon - T\frac{dS}{dt}}$$

where S is the entropy per unit mass.

Energy transport

In Lecture 5, we used dimensional arguments to show, using only the hydrostatic equilibrium equation and the perfect gas equation of state, that a star is much hotter at the center than it is near the surface. From thermodynamics, we know that heat flows from higher to lower temperatures, the more so as the temperature gradient increases. Thus, we expect that our fourth differential equation will relate the flow of energy to the temperature gradient: $dT/dr = f(L,\ldots)$. The form of this relationship will depend upon the dominant mode of energy transport. We'll find that when all the energy is transported by radiation,

$$\boxed{\frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4acT^3}\frac{L}{4\pi r^2}} \qquad \text{radiative transport}$$

where $\bar{\kappa}$ is a suitably frequency-averaged (Rosseland mean) opacity. An equation of similar form can be used when conduction is important, as in white dwarfs. This is not surprising, since optically thick radiative transport (photon "collisions") and heat conduction (particle collisions) are both diffusive, random-walk processes.

However, we will also show that, when the temperature gradient exceeds a critical value known as the adiabatic gradient, the dominant mode of energy transport switches from radiation to convection (think of water coming to a boil when heated strongly from below). Convection is so efficient that usually only a tiny “superadiabatic” temperature gradient is necessary to transport all the luminosity of a star, in which case, to an excellent approximation, the convective temperature gradient has the adiabatic value and

$$\boxed{\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{\rm ad} = -\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr}} \quad \text{convective transport}$$

where $\Gamma_2\,(\rho,T,\boldsymbol{X})$ coincides, for an ideal gas, with $\gamma \equiv C_p/C_v$, a ratio of specific heats. For a monatomic ideal gas, $\gamma = 5/3$. The adiabatic approximation to the convective temperature gradient is not good near the surface of a cool star, where a substantially superadiabatic gradient may be required to transport enough flux.

Only one of the two equations for dT/dr is used at a given place and time. If the gradient predicted by the radiative transport equation exceeds the adiabatic gradient, the convective equation applies. The convective gradient is still related to luminosity in the sense that a larger gradient transports more heat, but $(dT/dr)_{\rm ad} = f(L,\ldots)$ is a very weak function of L (or, conversely, the heat flux is a very strong function of the superadiabatic gradient).

Composition changes

Recall that X_Z is the fraction by mass of element Z: $X_Z = m_Z n_Z/\rho$. It is most convenient to use the Lagrangian description, $X_Z = X_Z\,(M,t)$, because then, if there is no particle diffusion between mass shells, only nuclear reactions can change X_Z. Then

$$\frac{dX_Z}{dt} = \frac{m_Z}{\rho}\frac{dn_Z}{dt}$$

Let r_{jk} give the rate, per unit volume and time, at which nuclear reactions transform nuclei of type j to type k. The concentration of nucleus Z is increased by reactions r_{jZ} that create Z and decreased by reactions r_{Zk} that destroy Z:

$$\boxed{\frac{dX_Z}{dt} = \frac{m_Z}{\rho}\left[\sum_{j=1}^{n} r_{jZ} - \sum_{k=1}^{n} r_{Zk}\right] \quad Z = 1, n}$$

where the sums are over all Z. In practice, the number n of transmuting elements that play a significant role in influencing stellar structure at any one time is usually small. However, if one is tracking elemental or isotopic abundances per se, n may need to be large.

This expression of the composition equations is purely formal, in that we know nothing as yet about the reaction rates $r_{jk}(\rho,T,\boldsymbol{X})$. In fact, we won't develop the composition equations further—our interest in the reaction rates will center on their role in the energy equation. However, it is important to understand that, although much can be learned from sequences of static stellar models (stellar *structure*), the composition equations are an essential ingredient of stellar *evolution*.

Timescales

We've previously introduced three important timescales—dynamical, thermal (Kelvin-Helmholtz), and nuclear—that are arranged in a hierarchy:

$$\tau_{\mathrm{dyn}} \ll \tau_{\mathrm{th}} \ll \tau_{\mathrm{nuc}}$$

In the Sun, for example, $\tau_{\mathrm{dyn}} \sim 1$ hr, $\tau_{\mathrm{dyn}} \sim 10^{6.5}$ y, $\tau_{\mathrm{nuc}} \sim 10^{10.5}$ y.

If the star changes on timescales much longer than τ_{dyn}, the momentum equation reduces to the hydrostatic (mechanical) equilibrium equation

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

With the exception of pulsating stars, we will not consider situations in which the time-dependent momentum equation is required.

If the star changes on timescales much longer than τ_{th}, the energy equation reduces to

$$\frac{dL}{dM} = \varepsilon$$

and the star is in thermal equilibrium. The combination of mechanical and thermal equilibrium is sometimes known as *complete* equilibrium. Unlike mechanical equilibrium, which is a good approximation for all but the most dynamic phases of stellar evolution, thermal equilibrium is *not* a good approximation during many evolutionary phases—for example, during pre-main sequence evolution when the star derives energy from gravitational contraction rather than nuclear burning.

The nuclear timescale is quite long—longer than the estimated age of the solar system. Therefore, we do not expect that the Sun is in nuclear equilibrium.

The equations of static stellar structure

In the approximation of complete equilibrium, the evolution equations separate into two groups: the structure equations, involving only spatial derivatives, and the composition equations, involving only time derivatives. Thus, if $\boldsymbol{X}(M)$ is specified at some time t, the structure equations form a complete set of ordinary differential equations that determine the spatial structure of the star. Here are the structure equations:

$r(M)$	$M(r)$	
$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$	$\frac{dM}{dr} = 4\pi r^2 \rho$	conservation of mass
$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$	$\frac{dP}{dr} = -\frac{GM}{r^4}\rho$	hydrostatic equilibrium
$\frac{dL}{dM} = \varepsilon$	$\frac{dL}{dr} = 4\pi r^2 \rho\, \varepsilon$	thermal equilibrium
$\frac{dT}{dM} = -\frac{3\bar{\kappa}}{4acT^3}\frac{L}{16\pi^2 r^4}$	$\frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4acT^3}\frac{L}{4\pi r^2}$	radiative transport
or	*or*	*or*
$\frac{dT}{dM} = -\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dM}$	$\frac{dT}{dr} = -\frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\frac{dP}{dr}$	convective transport

Functional relationships

The structure equations contain material or "constitutive" functions that must be known in order to solve the equations:

$P = P(\rho, T, \boldsymbol{X})$	pressure
$\bar{\kappa} = \bar{\kappa}(\rho, T, \boldsymbol{X})$	mean opacity
$\varepsilon = \varepsilon(\rho, T, \boldsymbol{X})$	energy generation rate
$\Gamma_1 = \Gamma_1(\rho, T, \boldsymbol{X})$	first adiabatic exponent

We have studied the pressure equation of state for ideal and degenerate gases. Next we'll investigate the opacity, the adiabatic exponent, and the energy generation rate.

Boundary conditions

We require four boundary conditions for four first-order differential equations. The stellar structure problem is complicated by the fact that not all the boundary conditions can be imposed at one boundary; rather, they are split between the center and the surface of the star.

The central boundary conditions simply express the lack of a singularity there:

$$\text{At } r = 0: \quad M = 0 \text{ and } L = 0$$

If we specify the mass of the star, we can't prescribe P and T at the center; they are results of the model (conversely, if we do specify P and T at the center, the total mass is a result of the computation and cannot be independently chosen). Therefore, we look for conditions on the outer boundary.

The simplest outer conditions would be

$$\text{At } r = \mathcal{R}: \quad P = 0 \text{ and } T = 0$$

commonly known as the *zero boundary conditions*. The application of zero boundary conditions can give adequate models of the interior (although obviously not the photospheric surface!) for upper main sequence stars, which are radiative in their outer regions. For cooler stars, in which convection is important in the outer layers, a better approximation to the outer boundary conditions is necessary.