(online at www.astro.umd.edu/~drabin/)

*The clear light of truth***.**

We've now seen the basic equations of stellar evolution, including the equation expressing the radiative transport of luminosity:

$$\frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4acT^3}\frac{L}{4\pi r^2}$$

Today we'll derive this equation and explore the opacity $\bar{\kappa}$ that appears in it.

Micro–review

Linear aborption coefficient $dI_\nu = -k_\nu I_\nu \, ds \quad [k_\nu] = \mathrm{m}^{-1}$

Cross section $k_\nu = n\sigma_\nu \quad [\sigma_\nu] = \mathrm{m}^2$

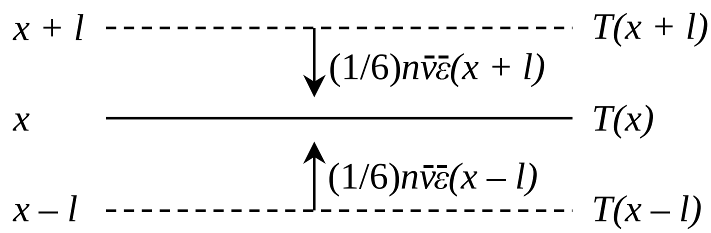
Mean free path $\ell_\nu = 1/n\sigma_\nu = 1/k_\nu$

Mass absorption coefficient $k_\nu = \rho\kappa_\nu \quad [\kappa_\nu] = \mathrm{m}^2 \ \mathrm{kg}^{-1}$

Thermal conduction

Heat flows from hot to cold. Experimentally, for a wide range of gases, liquids, and isotropic solids, the heat flux is found to be proportional to the temperature gradient: $F = -K\nabla T$, where *K* is the coefficient of thermal conductivity. The microscopic basis for conduction is the random motion of particles. We can estimate the form of *K* with a simple argument.

Let $\bar{\varepsilon}(x)$ be the mean energy of a particle at *x*. At any moment, roughly 1/6 of the particles are headed in each of the 6 directions ±*x*, ±*y*, ±*z*. Thus, per unit time and area, about $\frac{1}{6}n\bar{v}$ particles cross a plane of constant *x* from below, and a similar number from above. The particles crossing from below experienced their last collision, on average, a distance *l* below the plane, where *l* is the mean free path, and therefore carry a mean energy $\bar{\varepsilon}(x-l)$ across the plane. The net energy flux in the +*x* direction is



$$F = \frac{1}{6}n\bar{v}\left[\bar{\varepsilon}(x-l) - \bar{\varepsilon}(x+l)\right]$$
$$= \frac{1}{6}n\bar{v}\left[-2l\frac{d\bar{\varepsilon}}{dx}\right] = -\frac{1}{3}n\bar{v}l\frac{d\bar{\varepsilon}}{dT}\frac{dT}{dx}$$

We can write this as

$$F = -K\frac{dT}{dx}$$

where the coefficient of thermal conductivity is

$$K = \frac{1}{3}n\bar{v}\,l\,C$$

and $C = d\bar{\varepsilon}/dT$ is the specific heat per particle. For a monatomic ideal gas, $\bar{\varepsilon} = \frac{3}{2}kT$ and $C = \frac{3}{2}k$. For a gas of neutral atoms, we can estimate the mean free path using $l = 1/n\sigma$ and $\sigma \approx \pi a_0^2$ where a_0 is the Bohr radius (O&C §9.2). The collision cross section for an ionized gas is not so obvious.

What about a *photon* gas? Rewrite the estimate above for conductive flux as

$$F = -\frac{1}{3}n\bar{v}l\frac{d\bar{\varepsilon}}{dT}\frac{dT}{dx} = -\frac{1}{3}\left(n\frac{d\bar{\varepsilon}}{dT}\right)(\bar{v})(l)\frac{dT}{dx} = -\frac{1}{3}\left(\frac{du}{dT}\right)(c)\left(\frac{1}{\bar{\kappa}\rho}\right)\frac{dT}{dx}$$

where u is energy density and the wavelength averaged mass absorption coefficient $\bar{\kappa}$ (the *opacity*) is effectively defined by its correspondence with the mean free path.

Finally, for blackbody radiation, $u = aT^4$ where $a = 4\sigma / c$ is the radiation constant. Using this in du/dT above gives

$$F = -\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{dx}$$

In a spherical star, $F = L/4\pi r^2$, $(\nabla T)_r = dT/dr$ and we obtain the equation for radiative energy transport:

$$\frac{dT}{dr} = -\frac{3\bar{\kappa}\rho}{4acT^3}\frac{L}{4\pi r^2}$$

In view of our rough derivation of the coefficient of thermal conductivity, it is fortuitous that the equation comes out right, including the numerical factors. However, the approximate derivation here brings out the underlying unity of heat conduction, whether by massive particles or photons.

Rosseland mean opacity

(O&C §9.2)

As a homework problem (O&C Problem 9.16), you'll derive from the radiative transfer equation the result

$$F_\lambda = -\frac{c}{\kappa_\lambda \rho}\frac{dP_{\text{rad},\lambda}}{dr}$$

For an isotropic radiation field, the radiation pressure is (Lecture 1) $P_{\text{rad},\lambda} = \dfrac{4\pi}{3c} I_\lambda$.

Further assume that the radiation is blackbody, $I_\lambda = B_\lambda$. Then

$$F_\lambda = -\frac{c}{\kappa_\lambda \rho}\frac{4\pi}{3c}\frac{dB_\lambda}{dT}\frac{dT}{dr}$$

Integrating over all wavelengths,

$$F = -\frac{4\pi}{3\rho}\left(\int_0^\infty \frac{1}{\kappa_\lambda}\frac{dB_\lambda}{dT}\,d\lambda\right)\frac{dT}{dr}$$

Now, if we *define* the *Rosseland mean opacity* by

$$\frac{1}{\bar{\kappa}} = \int_0^\infty \frac{1}{\kappa_\lambda}\frac{dB_\lambda}{dT}\,d\lambda \Big/ \int_0^\infty \frac{dB_\lambda}{dT}\,d\lambda$$

we obtain

$$F = -\frac{4\pi}{3\bar{\kappa}\rho}\left(\frac{d}{dT}\int_0^\infty B_\lambda\,d\lambda\right)\frac{dT}{dr} = -\frac{4\pi}{3\bar{\kappa}\rho}\left(\frac{d}{dT}\frac{acT^4}{4\pi}\right)\frac{dT}{dr}$$
$$= -\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{dr}$$

which recovers our basic result while revealing the nature of the mean opacity. Thinking in terms of *transparency* ($1/\kappa_\lambda$) instead of absorption, the Rosseland mean transparency is the integral of the monochromatic transparency weighted by the temperature derivative of the Planck function. Regions, if any, of particularly high transparency (low opacity tend to dominate the integral. The weighting function de-emphasizes high and low frequencies.

Note that, when more than one source of opacity contributes significantly to the total opacity at some frequencies, the Rosseland mean of the total opacity is not in general equal to the sum of the individual Rosseland mean opacities. That is, if opacity sources $\kappa_1(\nu)$ and $\kappa_2(\nu)$ have Rosseland means $\bar{\kappa}_1$ and $\bar{\kappa}_2$, $\overline{\kappa_1 + \kappa_2} \neq \bar{\kappa}_1 + \bar{\kappa}_2$.

Sources of opacity
(O&C §9.2)

Bound-bound transitions

The absorption (or emission) of a photon when an electron makes an upward (or downward) transition between two bound energy levels is the familiar process that gives rise to discrete spectral lines. Although bound-bound opacity is most important in cool stars and in the outermost layers of hotter stars, it makes a nontrivial contribution to the Rosseland mean over a fairly large region of the ρ–T plane. In the Sun, the ratio of the total opacity with lines to that without can reach 2–3. There is no convenient formula that encapsulates bound-bound opacity.

Bound-free transitions

Bound-free absorption, also known as *photoionization*, occurs when an incident photon has enough energy to liberate an electron from a bound atomic state. If we consider the ionization of an electron from a particular bound level, the cross section is zero until it spikes sharply (*ionization edge*) at $h\nu = \chi$ (χ is the ionization potential), then drops to higher energies as ν^{-3}. Thus, the total opacity due to bound-bound transitions is a summation over sawtooth-like patterns followed by an integration over frequency. Kramers showed in a semi-classical approximation that

$$\bar{\kappa}_{\mathrm{bf}} \approx 10^{24.6} \frac{g_{bf}}{t} Z(1+X) \frac{\rho}{T^{3.5}} \ \ \mathrm{m^2\ kg^{-1}}$$

where the Gaunt factor g_{bf} is of order unity and the mysterious guillotine factor t is typically in the range [1,100]. The inverse process to photoionization is *radiative recombination*.

The negative hydrogen ion (H^-) is a dominant source of bound-free opacity in the outer layers of cooler stars. There is a bound state for a second electron in the field of a proton, but the binding energy of the second electron is only 0.75 eV, compared to 13.6 eV for the first electron. When hydrogen is predominantly neutral, metals ($Z > 2$), which often have first ionization potentials of only a few eV, provide most of the free electrons and therefore, through H^-, control the total opacity.

Free-free transitions

A free electron in the vicinity of an ion can accelerate and in so doing absorb or emit a photon: free-free absorption or free-free emission (also known as bremsstrahlung). Kramers came to the rescue here too with the approximate forumula

$$\bar{\kappa}_{\rm ff} \approx 10^{21.6} g_{ff} (1-Z)(1+X) \frac{\rho}{T^{3.5}} \ \text{ m}^2 \text{ kg}^{-1}$$

Note that $\bar{\kappa}_{\rm bf}$ and $\bar{\kappa}_{\rm ff}$ share the same functional dependence on ρ and T.

Electron scattering

The cross section for electron scattering may be calculated classically by considering that an electromagnetic wave passing an electron has an electric field that causes the electron to oscillate, which then tends to re-emit a photon of the same frequency in another direction. In terms of the classical electron radius $r_e = e^2 / m_e c^2$,

$$\kappa_{\rm es} = \frac{8\pi}{3} \frac{r_e^2}{\mu m_H} = 0.020(1+X) \ \text{ m}^2 \text{ kg}^{-1}$$

Note that $\kappa_{\rm es}$ does not depend on frequency, so $\bar{\kappa}_{\rm es} = \kappa_{\rm es}$. The formula assumes a fully-ionized, pure hydrogen-helium mixture, which is generally a good approximation where electron scattering is important. The cross section (and opacity) must be modified for relativistic effects at very high temperatures ($kT \gtrsim m_e c^2$) or if the electrons are significantly degenerate.

Electron conduction

Because normally the mean free path of a photon is much longer than the mean free path of an electron or ion, heat conduction by particles can be ignored relative to radiative diffusion. However, as we mentioned in our discussion of the Fermi-Dirac distribution, a highly degenerate electron gas becomes highly conductive because an electron on the surface of the Fermi sea cannot give up energy in a collision since all lower energy states are already occupied. Electron conduction is important in the cores of evolved stars and in white dwarfs.

Opacity in the ρ–T plane

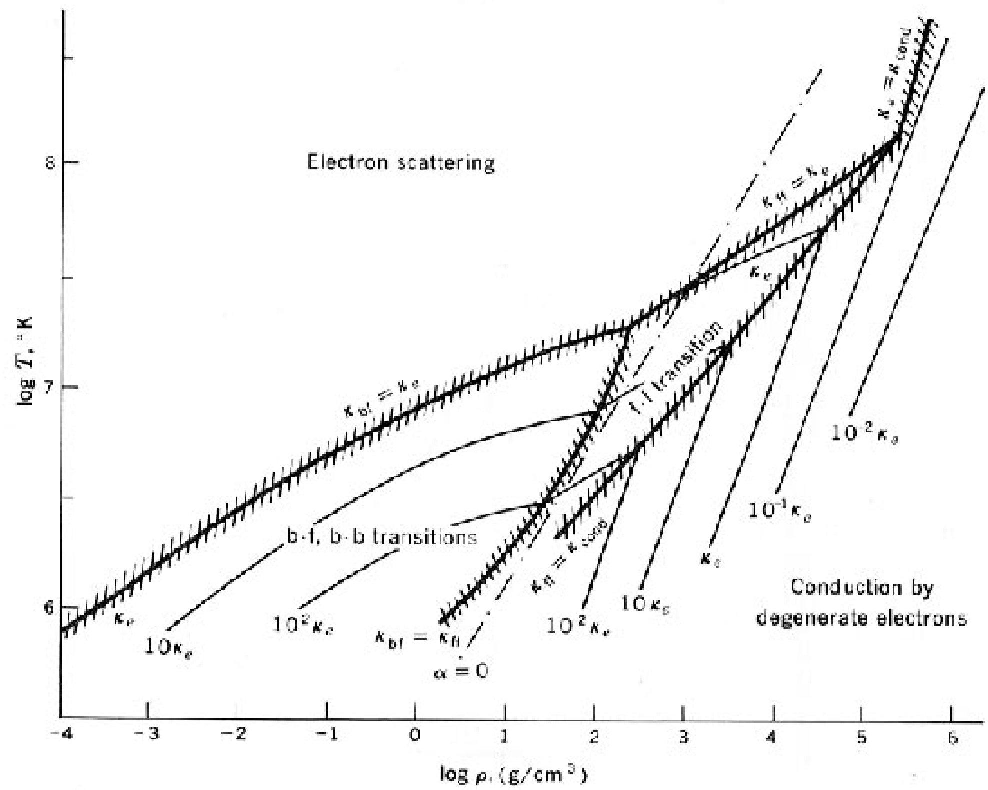


Fig. 3-15 The total opacity of population I composition. The ρT plane is divided into four domains according to which opacity source is the most important for energy transport, electron scattering, bound-free transitions, free-free transitions, and conduction by degenerate electrons (to be discussed in Sec. 3-4). The lines designating these boundaries are cross-hatched. Contours of constant opacity are labeled by the value of κ in terms of the opacity κ_e due to electrons. A dashed line shows where the degeneracy parameter $\alpha = 0$. (*After C. Hayashi, R. Hoshi, and D. Sugimoto, Progr. Theoret. Phys. Kyoto, Suppl.* 22, 1962.)

Rosseland mean opacity graphically

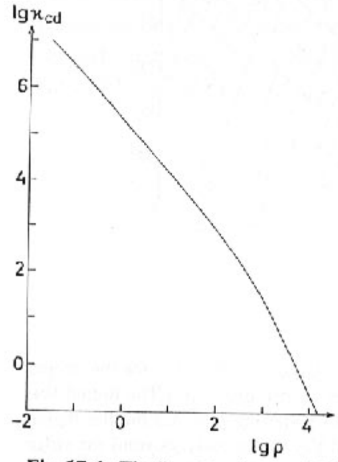


Fig. 17.4. The "conductive opacity" κ_{cd} (in $cm^2\ g^{-1}$) of a hydrogen gas at $T = 10^7$ K against the density ϱ (in $g\ cm^{-3}$). (After HUBBARD, LAMPE, 1969)

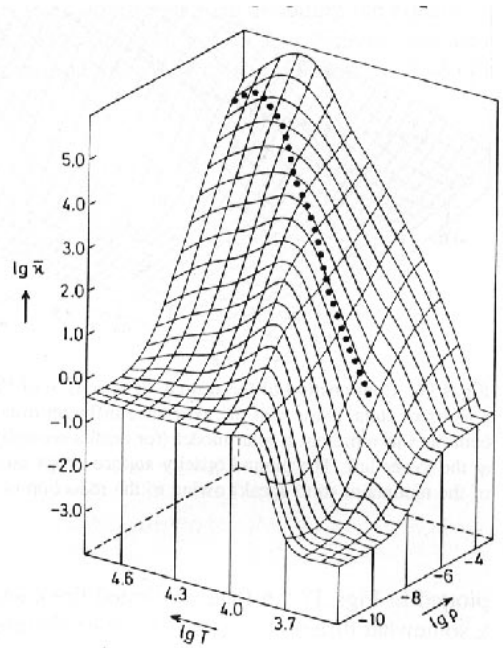


Fig. 17.5. The Rosseland mean of the opacity κ (in $cm^2\ g^{-1}$) as a function of ϱ (in $g\ cm^{-3}$) and T (in K) for a mixture with a hydrogen and helium content $X = 0.739$, $Y = 0.240$, respectively, according to calculations using the Los Alamos code for the outer layers of stars. The dotted line indicates a solar model, starting at the right end with the photosphere. The dominant absorption mechanisms at different parts of the model are discussed in the text. The continuation towards deeper regions is shown in Fig. 17.6

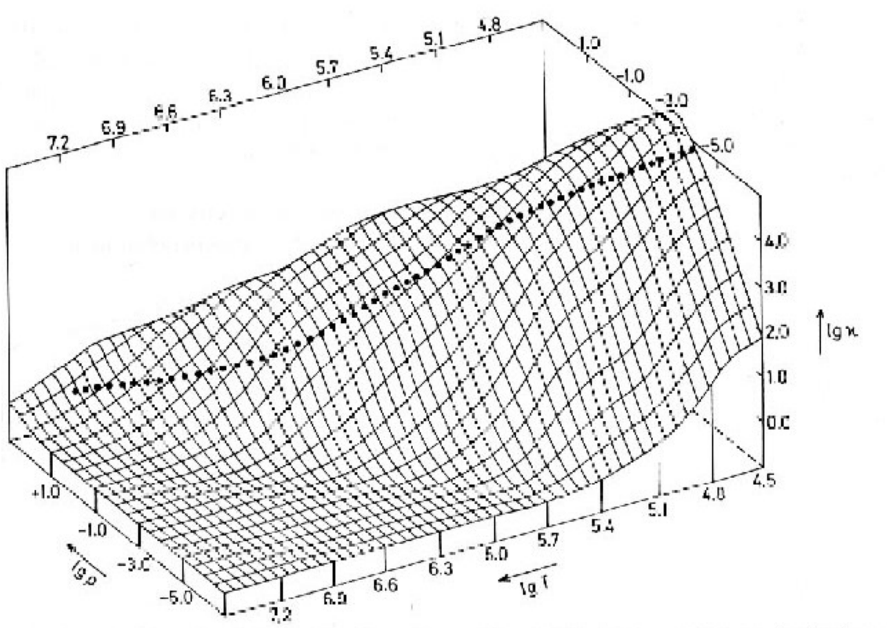


Fig. 17.6. Continuation of the display of opacity κ of Fig. 17.5 to larger ϱ and T, i.e. for the deeper regions of stars. (Note that the axes have different orientations in each illustration.) The dotted line continues to represent a solar model (for details see text). Electron scattering provides the flat region at the lower left. The plotted opacity surface drops away behind the visible part (beyond the ridge of the mountain, so to speak) owing to the reduction of effective opacity by conduction