ASTR120 Challenge Problem #3 – (Hamilton)
Optional, due by the end of the semester

In this problem you will determine the point of maximum brightness of an inner planet viewed from an outer planet. The calculations require algebra, geometry, trignonometry, calculus, and proper techniques of approximation and are a look ahead at the kind of problems that you will do in upper level Astrophysics classes.

Consider two planets on coplaner and circular orbits with orbital radii a_1 and a_2 and $a_1 < a_2$. Two effects will change the apparent brightness of the interior planet as seen from the exterior planet: i) the apparent area of the planet on the sky (which varies as the inverse square of its distance) and ii) the phase of the planet. As in the inner planet catches up to the outer planet, the first effect increases its brightness while the second effect decreases it. Is there a point along the orbit where the brightness is maximum? Where is it?

We need a function that gives the brightness B of the planet as a function of distance. Use $B \propto \cos^2(\Psi/2)/r^2$ where the numerator is the fraction of the planet's disk visible from the outer planet that is illuminated by sunlight, Ψ is the angle between the Sun and the outer planet as seen from the inner, and r is the distance between the planets.

a) Start by drawing a picture of the system, labeling key distances and angles. Check the cosine function to see if it behaves properly at $\Psi = 0°$, $\Psi = 90°$, $\Psi = 180°$. Now, calculate the brightness of Venus seen from Earth ($a_1 = 0.72a_2$) at three locations: i) furthest distance from us, ii) "half-Venus" phase, and iii) closest distance. Take $a_2 = 1$AU for simplicity. Repeat for Mercury ($a_1 = 0.4a_2$) and a hypothetical interior planet with ($a_1 = 0.2a_2$), gathering all results into a table. What patterns do you see in these brightnesses?

b) Now solve for the point along the orbit where the brightness is maximum for arbitrary a_1. Start by converting B to a pure function of r.

c) Now calculate r, $\cos^2(\Psi/2)$, and B for Mercury and Venus at the point of maximum brightness that you found in part b). Check you results against your table from part a).

d) Now calculate r, $\cos^2(\Psi/2)$, and B at the point of maximum brightness for close planet separations ($a_2 - a_1 = \delta$, in the limit of small δ). Use approximations valid for small δ and put all distances in terms of δ.

e) For what value of a_1 is the point of maximum brightness at full phase?

f) For what value of a_1 does the point of maximum brightness occur when the planet is 1/2 full?

g) Finally, sketch your summary plot, pencil on paper, showing how full the planet is at maximum brightness on the y-axis vs. a_1/a_2 on the x-axis. Include your results from parts c)-f). Where should Earth be to be as bright as possible as seen from Jupiter?