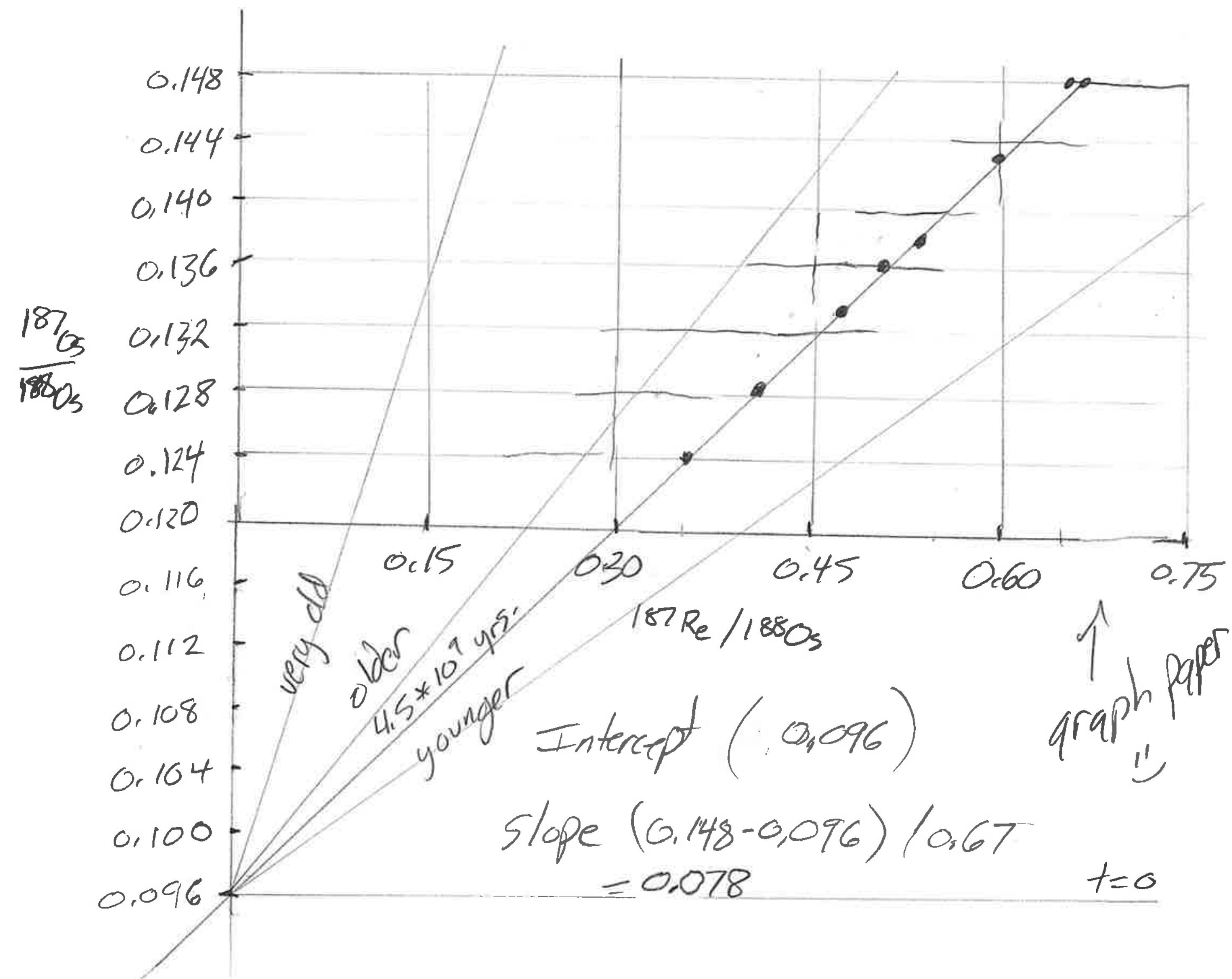
ASTR 430 Homework #5
Solution Set

Hamilton

Problem 11-10

a) b) c) Age Dating Meteorites using $^{187}Re \rightarrow {}^{187}Os$ w/ $t_{1/2} = 4.16 \times 10^{10}$ yrs.



Intercept (0.096)

Slope $(0.148 - 0.096) / 0.67 = 0.078$

Now, see Fig 11.18 and use the equation given there for slope

$$\text{slope} = e^{(t-t_0)/t_m} - 1$$

where $\Delta T = t - t_0$ = elapsed time

Eq. 11.5 → $t_{1/2} = \ln 2 \, t_m = 4.16 * 10^{10}$ yrs.

$\Rightarrow t_m = 6.00 * 10^{10}$ yrs.

$$0.078 = e^{\Delta t / 6.0 * 10^{10}} - 1$$

$$\frac{\Delta t}{6.0 * 10^{10}} = \ln(0.078 + 1) = 0.075$$

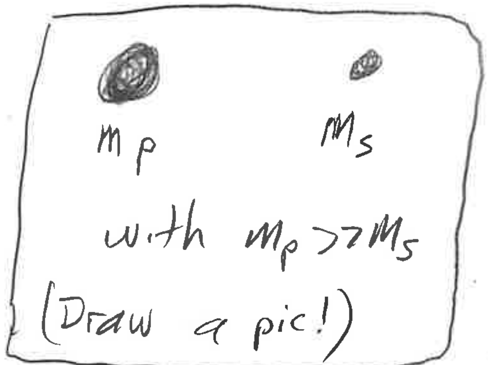
$$\Rightarrow \boxed{\Delta t = 4.5 * 10^{9} \text{ yrs.}}$$

Initial Abundance of $\boxed{\frac{^{187}Os}{^{188}Os} = 0.096}$

(y-intercept)

Problem 13-1

Hill Sphere vs. Roche Limit



These concepts are related. The Hill sphere (radius R_H) is the region around m_s in which satellites can stably orbit.

The Roche Limit (distance r_R) is the distance from m_p at which m_s can be tidally disrupted. In each case, forces from m_p are disrupting the m_s system. The force doing the disrupting in each case is the tidal force (= m_p gravity + centrifugal force).

a) Derive 13.7 from 13.6

$$\frac{a}{R_p} = 3^{1/3}\left(\frac{\rho_p}{\rho_s}\right)^{1/3}$$

$$\frac{GM_s}{R_s^2} = \frac{3GM_pR_s}{a^3}$$

solve 13.6 for a^3 and use
$M_p = 4/3\,\pi R_p^3 \rho_p$ and $M_s = 4/3\,\pi R_s^3 \rho_s$

$$\Rightarrow a^3 = 3R_s^3\left(\frac{M_p}{M_s}\right) = 3R_s^3\,\frac{R_p^3\rho_p}{R_s^3\rho_s}$$

$$\boxed{\left(\frac{a}{R_p}\right) = 3^{1/3}\left(\frac{\rho_p}{\rho_s}\right)^{1/3}}$$ ✓

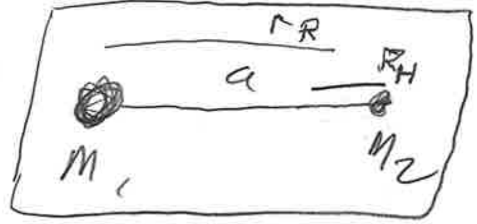
13.7

b) Derive 13.7 from 2.28

$$R_H = \left(\frac{m_2}{3(m_1+m_2)}\right)^{1/3} a$$

Note that the a's are formally different $a = r_R$ from a) is distance from m_1 to the limit. a from b) is the

distance between the two masses.
But not an issue since $a >> R_H$.



Now we know we need to get rid of masses - that's easy so ~~start~~ there.

$$\frac{m_2}{m_1 + m_2} = \frac{\rho_2 R_2^3}{\rho_1 R_1^3 + \rho_2 R_2^3} \approx \frac{\rho_2 R_2^3}{\rho_1 R_1^3}$$

where we use $m_1 >> m_2$

so $$\left(\frac{a}{R_H}\right)^3 = 3\left(\frac{\rho_1}{\rho_2}\right)\frac{R_1^3}{R_2^3}$$

$$\underbrace{\left(\frac{a}{R_H}\right)\left(\frac{R_2}{R_1}\right)} = 3^{1/3}\left(\frac{\rho_1}{\rho_2}\right)^{1/3}$$ ← RHS of 13.7

$\rho_1 = \rho_p$; $\rho_2 = \rho_s$

This equals a/R_p if $R_2 = R_H$ ← Is this justified?

$R_2 = R_H$ basically says that if the satellite fills its Hill sphere, then it will be disrupted because the edge of the satellite reaches the Roche limit. ✓

Done! :)

Problem 15-3 Interstellar Gas Cloud

a) On a circular orbit at 1AU, an H_2 molecule has Energy

$$E = \frac{1}{2} m_H v^2 - \frac{GMm_H}{r} = -\frac{GMm_H}{2a}$$

where M = the Sun's mass
m_H = mass of H_2 molecule
v = velocity
r = distance from the Sun
a = semimajor axis

So the total energy is

$$E = -\frac{(6.67 \times 10^{-11})(2 \times 10^{30}\,kg)(2 \cdot 1.67 \times 10^{-27}\,kg)}{2(1.496 \times 10^{11}\,m)}$$

$$= -1.49 \times 10^{-18}\,J$$

The velocity of the H_2 molecule along its circular orbit will be:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\frac{GM}{r}} = 2.99 \times 10^{4}\,m/s$$
$$= 29.9\,km/s$$

b) This is clearly a conservation of Energy problem. We balance the energy of the H_2 molecule before it drops down to 1 AU agaist the energy afterwards. In symbols:

$$(1) \quad E_{before} = E_{orbit} + E_{heat}$$

We can find E_{before} from

$$E = \frac{1}{2} m_H v^2 - \frac{G M m_H}{r} = 0 \qquad \text{when } \begin{matrix} r = \infty \\ v = 0 \end{matrix}$$

So (1) becomes

$$0 = -1.49 * 10^{-18} \, J + E_{heat}$$

$$E_{heat} = 1.49 * 10^{-18} \, J$$

From Physics, $E_{heat} = \frac{3}{2} kT$ where k is Boltzmann's constant $k = 1.38 * 10^{-23} \, J/°K$

so $$T = \frac{2}{3k} \cdot (1.49 * 10^{-18} \, J)$$

Hot! $$\boxed{T = 72{,}000 \, °K}$$

This is the primary source of heat for the solar nebula, the disk of mostly Hydrogen and Helium that circled the protosun. Most of the energy released by freefall escapes as radiation, but a large amount is left over for the disk. Since more energy is released as gas falls further in toward the central mass concentration, we expect that the solar nebula's temperature will increase as we approach the protostar.

Later, fusion starts in the proto star, and another source of heat is tapped.

Problem 15-12

Find smallest body melted by decay of ^{26}Al

Heating rate initially: $10^{-3} \frac{J}{m^3 s}$

$\tau_m^{-1} = 8 \times 10^{-7}\ yr^{-1}$

$c_p = 700\ J/_{kg\,K}$

diffusivity $10^{-6}\ m^2/_s$

$T_{melt} = 1800\ K$

Following the hint, we calculate timescales for 1) heat to diffuse and 2) To raise T to T_{melt}

1) Diffusivity - no help from the book :(

wikipedia: Diffusion Equation

is $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$ w/ α = diffusivity

Looks like this is a rough estimation problem, why? We ignore shape and composition of the asteroid, its initial Temperature, etc.

So diffusion timescale is, dimensionally,

$$\text{time} = \frac{(\text{length})^2}{\alpha}$$

The heating timescale is just

$$\tau_m = 1/8*10^{-7}\,\text{yr.} = 1.25*10^{6}\,\text{yr.}$$
$$= 3.9*10^{13}\,\text{s}$$

Equate!

$$\frac{r^2}{10^{-6}} = 3.9*10^{13} \Rightarrow r = \sqrt{3.9*10^{7}}$$

$r = 6240\,m \;=\; 6.2\,km$

$$\boxed{\text{so for Asteroids} > 6.2\,km\text{, the heat builds up faster than it diffuses away.}}$$

Done? Not quite. We have to check whether enough heat is generated to melt the body.

$$\text{so} \quad H \approx \dot{H}\,\tilde{\tau}_m = c_p\, M \cdot dT$$

estimate of total heating

Temperature rise

check if $dT > 1800\,K$

$$dT = \frac{\dot{H}\,\tilde{\tau}_m}{c_p\, M} \;; \quad \dot{H} = 10^{-3}\,\frac{J}{m^3 s} \text{ (Volume)}$$

$m = \rho\,(\text{Volume})$

$\uparrow$ —— estimate at $3000\ \mathrm{kg/m^3}$

$$\Rightarrow\ dT = \frac{10^{-3}\ \mathrm{J/m^3 s}\ (\text{Volume})\ 3.9*10^{13}\ \mathrm{s}}{\left(700\ \mathrm{J/kg\,K}\right)\ 3000\ \mathrm{kg/m^3}\ (\text{Volume})}$$

$$= \frac{3.9*10^{10}}{2.1*10^{6}} = \boxed{1.9*10^{4}\ \mathrm{K}\quad \text{hot!}}$$

plenty of energy in decay of ^{26}Al to melt rock + iron.