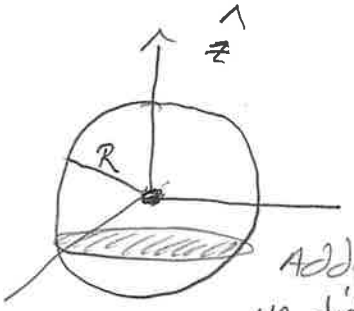
Moment of Inertia of a uniform sphere - cylindrical coordinates

$$I = \int dI = \int_M r_{xy}\, dM$$



Adding up disks to make a sphere

$$= \int_{-R}^{R}\int_{0}^{r}\int_{0}^{2\pi} r_{xy}^2\, \rho\, r_{xy}\, d\theta\, dr_{xy}\, dz$$

$$= 2\pi\rho \int_{-R}^{R} \frac{r^4}{4}\, dz \quad \longleftarrow \quad \text{using } r^2 = R^2 - z^2 \text{ and expanding } (R^2 - z^2)^2$$

$$= \frac{\rho\pi}{2}\left(R^4 z - \frac{2R^2 z^3}{3} + \frac{z^5}{5}\Big|_{-R}^{R} \right)$$

$$= \frac{\rho\pi}{2}\left[2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5} \right]$$

$$= \rho\pi R^5 \left[1/3\, R^5 + R^5/5 \right]$$

$$= \boxed{\frac{8\pi}{15}\rho R^5 = I}$$

using $M = \frac{4}{3}\pi R^3 \rho$

$$\Rightarrow \boxed{I = \frac{2}{5} M R^2}$$