1. a) Use a symmetry argument and dimensional analysis to determine how the direction and magnitude of gravity from Saturn’s wide flat ring (modeled as a uniform-density flat sheet of infinite extent) varies with height above the ring.

b) Show that the undetermined constant from dimensional analysis is $2\pi$ using Gauss’ Law.

c) Find the vertical component of gravity from Saturn. Compare this with your answer for the ring’s gravity. At what height(s) are the two equal? Plug in numbers for Saturn’s ring assuming that its total mass of $M \approx 6.5 \times 10^{-8} M_p$ is spread uniformly between radii $1.5 R_s$ and $2.3 R_s$ ($M_p$ is the planet’s mass). How does the transition height compare to the $\approx 100m$ thickness of Saturn’s rings?

d) Now imagine a ring “atmosphere” - gas molecules above and below the rings of Saturn. There is some evidence that a ring atmosphere of Oxygen, Hydrogen, Water Vapor, and OH actually exists. Use previous handouts and homeworks to help you derive how the density of gas should vary as a function of height above the rings. Be sure to use the FULL gravity (ring + vertical component of Saturn).

2. Consider the 2:1 resonance between Io and Europa which is governed by the following approximate expressions:

$$\frac{da_I}{dt} = 2\beta m_E a_I^2 n_I e_I \sin(\Psi) + \dot{a}_{\text{tides}} \quad \frac{da_E}{dt} = -4\beta m_I a_E^2 n_E e_I \sin(\Psi)$$

$$\frac{de_I}{dt} = -\beta m_E a_I^2 n_I \sin(\Psi) + \dot{e}_{\text{tides}} \quad \frac{de_E}{dt} = 0$$

$$\frac{d\varpi_I}{dt} = -\frac{\beta m_E a_I n_I}{e_I} \cos(\Psi) \quad \frac{d\varpi_E}{dt} = 0$$

where the subscripts $E$ and $I$ stand for Europa and Io, respectively, $t$ is time, $\beta$ is a constant, $m$ is mass, $a$ is the semimajor axis, $n = \sqrt{(GM/a^3)}$ is the mean motion (the satellite’s average angular speed), $e$ is eccentricity, and $\varpi$ is the longitude of pericenter. The resonant angle $\Psi = \lambda_I - 2\lambda_E + \varpi_I$, $\lambda$ is the satellite longitude, and $d\lambda/dt \approx n$. Finally, the terms $\dot{a}_{\text{tides}}$ and $\dot{e}_{\text{tides}}$ are constants that represent the weak effects of tides.

a) Show that in the absence of tides ($\dot{a}_{\text{tides}} = \dot{e}_{\text{tides}} = 0$), the equations conserve energy as they must. You can use $2n_E \approx n_I$.

b) Now, because of their steep dependence on distance, the tides pushing Io outward dominate the tides pushing Europa outward. Thus, to a good approximation, we include the constant tidal terms $\dot{a}_{\text{tides}} > 0$ and $\dot{e}_{\text{tides}} < 0$ only in the Io equations. Set the full $da_I/dt$ equation equal to zero (not a great approximation since in resonance Io and Europa are still moving outward, but not too bad), solve for $\sin(\Psi)$ and use it to eliminate $\sin(\Psi)$ in $de_I/dt$. Show that $de_I/dt$ is positive for small $e_I$ and find the equilibrium eccentricity. The existence of this equilibrium eccentricity allows tides to dissipate energy in Io.

3. CHALLENGE PROBLEM (Extra Credit). Extend problem 1 from HW#6 to wide rings.

a) Write down and solve integral expressions for the total mass $M$, the orbital energy $E$, and
the orbital angular momentum $L$ of a vertically-thin ring of constant surface mass density $\sigma$, inner radius $r_1$, and outer radius $r_2$.

b) Use Lagrange Multipliers to find the extrema in $E$ subject to constant $M$ and constant $L$. Interpret and discuss your results.