6

The Disturbing Function

Do I dare
Disturb the universe?

T.S. Eliot, *The Love Song of J. Alfred Prufrock.*

6.1 INTRODUCTION

In §3 we approached the three body problem from the point of view of the location and stability of equilibrium points in the restricted problem. However, we made no attempt to tackle the more general problem of the motion of a third body under the gravitational effects of the two other bodies for arbitrary initial conditions. This problem is non-integrable, but we can make some progress by analysing the accelerations experienced by the three bodies. If their motions are dominated by a central, or primary body, then the orbits of the secondary bodies are conic sections with small deviations due to their mutual gravitational perturbations. In this chapter, we show how these deviations can be calculated by defining and analysing the *disturbing function.*

Consider a mass m_i orbiting a primary of mass m_c in an elliptical path. As we have seen in §2, this problem is integrable and the orbital elements, a_i, e_i, I_i, ϖ_i and Ω_i of the mass m_i are constant, provided the gravitational effect of the central body can be treated as arising from a point mass. If we now introduce a third mass, m_j, then the mutual gravitational force between the masses m_i and m_j results in accelerations in addition to the standard two body accelerations due to m_c. These additional accelerations of the secondary masses *relative* to the primary can be obtained from the gradient of a scalar function called the disturbing function.

This chapter is concerned with a mathematical analysis of the properties of a Fourier series expansion of the disturbing function. We show how particular problems in Solar System dynamics can be tackled by isolating

the appropriate terms in the expansion of the disturbing function and by assuming that the time-averaged contributions to the equations of motion of all the other terms are negligible. An understanding of the properties of the disturbing function is the key to understanding the dynamics of resonance and other long-period motions in the Solar System.

6.2 THE DISTURBING FUNCTION

Let the position vectors with respect to a fixed origin, O, of the three bodies of masses m_c, m_i and m_j be $\boldsymbol{R}_c$, $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$ respectively. Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ denote the position vectors of the secondary masses m_i and m_j relative to the primary where

$$|\boldsymbol{r}_i| = r_i = \left(x_i^2 + y_i^2 + z_i^2\right)^{1/2} \qquad |\boldsymbol{r}_j| = r_j = \left(x_j^2 + y_j^2 + z_j^2\right)^{1/2} \tag{6.1}$$

and

$$|\boldsymbol{r}_j - \boldsymbol{r}_i| = \left[(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2\right]^{1/2} \tag{6.2}$$

and the primary is the origin of the coordinate system (see Fig.6.1).

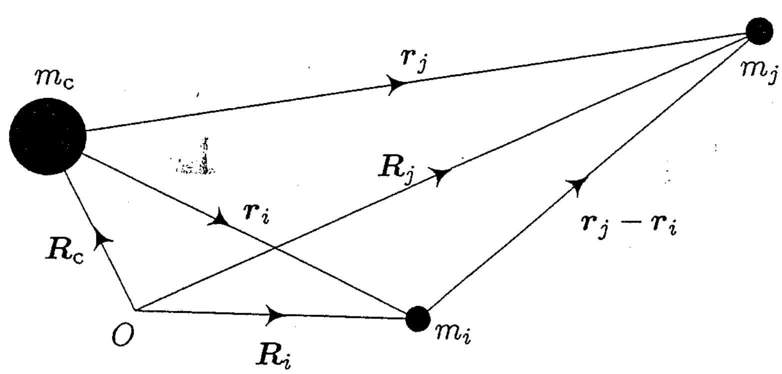

Figure 6.1 The position vectors, $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ of two masses, m_i and m_j, with respect to the central mass, m_c. The three masses have position vectors $\boldsymbol{R}$, $\boldsymbol{R}'$ and $\boldsymbol{R}_c$ with respect to an arbitrary, fixed origin, O.

From Newton's laws of motion and the law of gravitation we obtain the equations of motion of the three masses in the inertial reference frame:

$$m_c \ddot{\boldsymbol{R}}_c = \mathcal{G} m_c m_i \frac{\boldsymbol{r}_i}{r_i^3} + \mathcal{G} m_c m_j \frac{\boldsymbol{r}_j}{r_j^3} \tag{6.3}$$

$$m_i \ddot{\boldsymbol{R}}_i = \mathcal{G} m_i m_j \frac{(\boldsymbol{r}_j - \boldsymbol{r}_i)}{|\boldsymbol{r}_j - \boldsymbol{r}_i|^3} - \mathcal{G} m_i m_c \frac{\boldsymbol{r}_i}{r_i^3} \tag{6.4}$$

$$m_j \ddot{\boldsymbol{R}}_j = \mathcal{G} m_j m_i \frac{(\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} - \mathcal{G} m_j m_c \frac{\boldsymbol{r}_j}{r_j^3}. \tag{6.5}$$

The accelerations of the secondaries relative to the primary are given by

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{R}}_i - \ddot{\boldsymbol{R}}_c \tag{6.6}$$

$$\ddot{\boldsymbol{r}}_j = \ddot{\boldsymbol{R}}_j - \ddot{\boldsymbol{R}}_c \tag{6.7}$$

Substituting the expressions for $\ddot{\boldsymbol{R}}_c$, $\ddot{\boldsymbol{R}}_i$ and $\ddot{\boldsymbol{R}}_j$ from (6.3)–(6.5) we have

$$\ddot{\boldsymbol{r}}_i + \mathcal{G}\left(m_c + m_i\right)\frac{\boldsymbol{r}_i}{r_i^3} = \mathcal{G}m_j\left(\frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{\left|\boldsymbol{r}_j - \boldsymbol{r}_i\right|^3} - \frac{\boldsymbol{r}_j}{r_j^3}\right) \tag{6.8}$$

$$\ddot{\boldsymbol{r}}_j + \mathcal{G}\left(m_c + m_j\right)\frac{\boldsymbol{r}_j}{r_j^3} = \mathcal{G}m_i\left(\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|^3} - \frac{\boldsymbol{r}_i}{r_i^3}\right). \tag{6.9}$$

These relative accelerations can be written as gradients of scalar functions, that is, we can write

$$\begin{aligned}\ddot{\boldsymbol{r}}_i &= \nabla_i\left(U_i + \mathcal{R}_i\right)\\ &= \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x_i} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y_i} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z_i}\right)\left(U_i + \mathcal{R}_i\right)\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}\ddot{\boldsymbol{r}}_j &= \nabla_j\left(U_j + \mathcal{R}_j\right)\\ &= \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x_j} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y_j} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z_j}\right)\left(U_j + \mathcal{R}_j\right)\end{aligned} \tag{6.11}$$

where

$$U_i = \mathcal{G}\frac{(m_c + m_i)}{r_i} \quad \text{and} \quad U_j = \mathcal{G}\frac{(m_c + m_j)}{r_j} \tag{6.12}$$

are the central, or two body parts of the total potential. The subscript i or j is included in the ∇ operator to emphasise that the gradient is with respect to the coordinates of the mass m_i. The $\mathcal{R}$ term in the potential is the disturbing function which represents the potential which arises from the other secondary mass. Since $\boldsymbol{r}_i$ is not a function of x_j, y_j and z_j and $\boldsymbol{r}_j$ is not a function of x_i, y_i and z_i, we can write

$$\mathcal{R}_i = \frac{\mathcal{G}m_j}{\left|\boldsymbol{r}_j - \boldsymbol{r}_i\right|} - \mathcal{G}m_j\frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_j^3} \tag{6.13}$$

$$\mathcal{R}_j = \frac{\mathcal{G}m_i}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|} - \mathcal{G}m_i\frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_i^3} \tag{6.14}$$

The leading terms in these expressions are called the *direct terms* while the other terms that arise from the choice of the origin of the coordinate system, are called the *indirect terms*. If the origin of the coordinate system was at the center of mass, then these indirect terms would not appear.

The above analysis can be extended to any number of bodies. In addition, the accelerations associated with the disturbing function can arise from any source and not just from point-mass gravitational forces. They could, for example, arise from a potential associated with the oblateness of the central mass (see §6.9). However, in what follows in this chapter we are mostly concerned with the particular case of two point-mass secondaries of masses m and m' and position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ relative to the central mass, where $r < r'$ always. With this notation, the equation of motion of the inner secondary is

$$\ddot{\boldsymbol{r}} + \mathcal{G}\left(m_\mathrm{c} + m\right)\frac{\boldsymbol{r}}{r^3} = \mathcal{G}m'\left(\frac{\boldsymbol{r}' - \boldsymbol{r}}{\left|\boldsymbol{r}' - \boldsymbol{r}\right|^3} - \frac{\boldsymbol{r}'}{r'^3}\right) \tag{6.15}$$

and its disturbing function can be written

$$\mathcal{R} = \frac{\mu'}{\left|\boldsymbol{r}' - \boldsymbol{r}\right|} - \mu'\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r'^3} \tag{6.16}$$

where $\mu' = \mathcal{G}m'$ and the associated reference orbit has osculating elements $n^2a^3 = \mathcal{G}\left(m_\mathrm{c} + m\right)$. Similar equations can be written for the outer secondary giving

$$\ddot{\boldsymbol{r}}' + \mathcal{G}\left(m_\mathrm{c} + m'\right)\frac{\boldsymbol{r}'}{r'^3} = \mathcal{G}m\left(\frac{\boldsymbol{r} - \boldsymbol{r}'}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} - \frac{\boldsymbol{r}}{r^3}\right). \tag{6.17}$$

The corresponding disturbing function for the outer secondary is then

$$\mathcal{R}' = \frac{\mu}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} - \mu\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} \tag{6.18}$$

where $\mu = \mathcal{G}m$ and the associated reference orbit has osculating elements $n'^2a'^3 = \mathcal{G}\left(m_\mathrm{c} + m'\right)$.

Although this is the most straightforward way to derive expressions for $\mathcal{R}$ and $\mathcal{R}'$, it is worth pointing out that this procedure and the resulting expressions are not unique. For example, it is possible to add an additional term, $\mathcal{G}m\boldsymbol{r}'/r'^3$, to each side of the equation of motion for the mass m', (6.17), resulting in a additional term $-\mu/r'$ in the expression for $\mathcal{R}'$; however, this requires that the associated reference orbit for m' has osculating elements $n'^2a'^3 = \mathcal{G}\left(m_\mathrm{c} + m + m'\right)$.

6.3 EXPANSION USING LEGENDRE POLYNOMIALS

Consider the configuration shown in Fig.6.2 where $\boldsymbol{r}$ and $\boldsymbol{r}'$ denote the position vectors of the masses m and m' respectively. Let ψ denote the angle between the two position vectors. From the cosine rule we have

$$\left|\boldsymbol{r}' - \boldsymbol{r}\right|^2 = r^2 + r'^2 - 2rr'\cos\psi \tag{6.19}$$

or, alternatively,

$$\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \frac{1}{r'}\left[1 - 2\frac{r}{r'}\cos\psi + \left(\frac{r}{r'}\right)^2\right]^{-1/2}. \tag{6.20}$$

This can be expanded in Legendre polynomials to give

$$\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) \tag{6.21}$$

where $P_0(\cos\psi) = 1$, $P_1(\cos\psi) = \cos\psi$, $P_2(\cos\psi) = \frac{1}{2}(3\cos^2\psi - 1)$, etc.

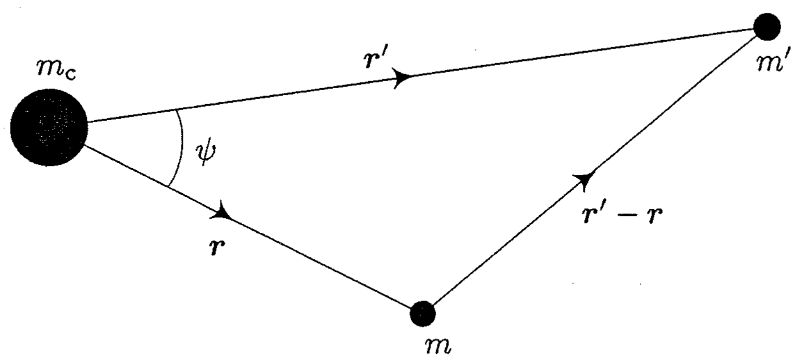

Figure 6.2 The position vectors, $\boldsymbol{r}$ and $\boldsymbol{r}'$ of two masses, m and m', with respect to a central mass, m_c. The angle between the position vectors is ψ.

Since $\boldsymbol{r} \cdot \boldsymbol{r}' = rr'\cos\psi = rr'P_1(\cos\psi)$, the disturbing function for the inner secondary can be written

$$\mathcal{R} = \frac{\mu'}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) \tag{6.22}$$

where the $P_0(\cos\psi)$ term has been omitted because it does not depend on r and, ultimately, we are only interested in the gradient of $\mathcal{R}$ with respect to the coordinates of the inner secondary. Similarly the disturbing function for the outer secondary can be written

$$\mathcal{R}' = \frac{\mu}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) + \mu\frac{r}{r'^2}\cos\psi - \mu\frac{r'}{r^2}\cos\psi. \tag{6.23}$$

Thus, apart from two extra terms (that are actually unimportant for the applications discussed in the book), the expressions for $\mathcal{R}$ and $\mathcal{R}'$ are very similar.

This chapter is concerned with the series expansion of the disturbing functions $\mathcal{R}$ and $\mathcal{R}'$ in terms of the orbital elements (as opposed to the

Cartesian coordinates) of m and m'. We use the standard orbital elements a, e, I, ϖ, Ω and λ to denote the semimajor axis, eccentricity, inclination, longitude of pericentre, longitude of ascending node and mean longitude, respectively, of the mass m, with similar primed quantities for the mass m'. We show that the expansion of $\mathcal{R}$ has the form

$$\mathcal{R} = \mu' \sum S(a, a', e, e', I, I') \cos \varphi \tag{6.24}$$

where φ is a permitted linear combination with general form

$$\varphi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega \tag{6.25}$$

where the j_i, $(i = 1, 2, \ldots, 6)$ are integers, and

$$\sum_{i=1}^{6} j_i = 0. \tag{6.26}$$

By knowing the explicit form of the function S and the permissible values of φ, we can identify those terms that make the dominant contributions to the equations of motion and, conversely, those that can be neglected.

To illustrate the nature of this expansion let us consider the special case where the orbits of the two masses m and m' lie in the same plane and we can ignore any terms arising from the inclination. In this case we can write the angle ψ as the difference of the true longitudes,

$$\psi = (f' + \varpi') - (f + \varpi) \tag{6.27}$$

where f and f' denote the true anomalies of m and m'. Hence,

$$\begin{aligned}\cos\psi &= \cos(f' + \varpi')\cos(f + \varpi) + \sin(f' + \varpi')\sin(f + \varpi) \\ &= (\cos f' \cos\varpi' - \sin f' \sin\varpi')(\cos f \cos\varpi - \sin f \sin\varpi) \\ &\quad + (\sin f' \cos\varpi' + \cos f' \sin\varpi')(\sin f \cos\varpi + \cos f \sin\varpi).\end{aligned}$$

We have already given series expansions for $\cos f$ and $\sin f$ in §2.5 and we can find similar series for $\cos f'$ and $\sin f'$ by substituting M' for M and e' for e. Taking these expansions to second degree in e and e' we find

$$\begin{aligned}\cos\psi = &\left(1 - e^2 - e'^2\right)\cos[M - M' + \varpi - \varpi'] \\ &- e\cos[M' - \varpi + \varpi'] - e'\cos[M + \varpi - \varpi'] \\ &+ e\cos[2M - M' + \varpi - \varpi'] + e'\cos[M - 2M' + \varpi - \varpi'] \\ &- \frac{1}{8}e^2\cos[M + M' - \varpi + \varpi'] - \frac{1}{8}e'^2\cos[M + M' + \varpi - \varpi'] \\ &+ \frac{9}{8}e^2\cos[3M - M' + \varpi - \varpi'] + \frac{9}{8}e'^2\cos[M - 3M' + \varpi - \varpi'] \\ &+ ee'\cos[\varpi - \varpi'] + ee'\cos[2M - 2M' + \varpi - \varpi'] \\ &- ee'\cos[2M + \varpi - \varpi'] - ee'\cos[2M' - \varpi + \varpi'].\end{aligned}$$

Even at this stage some properties of the expression for $\cos\psi$ are evident. It is clear that the degree of the eccentricity term associated with each cosine argument is *at least* the modulus of the sum of the coefficients of the mean anomalies in the argument. Another property shows up if we express the angles in terms of the mean longitudes rather than the mean anomalies using the substitutions $M = \lambda - \varpi$ and $M' = \lambda' - \varpi'$. This gives

$$\begin{aligned}\cos\psi = &\left(1 - e^2 - e'^2\right)\cos[\lambda - \lambda'] \\ &- e\cos[\lambda' - \varpi] - e'\cos[\lambda - \varpi'] \\ &+ e\cos[2\lambda - \lambda' - \varpi] + e'\cos[\lambda - 2\lambda' + \varpi'] \\ &- \frac{1}{8}e^2\cos[\lambda + \lambda' - 2\varpi] - \frac{1}{8}e'^2\cos[\lambda + \lambda' - 2\varpi'] \\ &+ \frac{9}{8}e^2\cos[3\lambda - \lambda' - 2\varpi] + \frac{9}{8}e'^2\cos[\lambda - 3\lambda' + 2\varpi'] \\ &+ ee'\cos[\varpi - \varpi'] + ee'\cos[2\lambda - 2\lambda' - \varpi + \varpi'] \\ &- ee'\cos[2\lambda - \varpi - \varpi'] - ee'\cos[2\lambda' - \varpi - \varpi']\end{aligned}$$

With this choice of angles it is clear that the sum of the integer coefficients of the longitudes in each argument is zero. This particular property is also true of the final expansion when the angles are expressed in terms of longitudes and it allows us to determine the permissible arguments.

If we now turn our attention to the radially dependent parts of the disturbing function (6.22), we can write

$$\mathcal{R} = \frac{\mu'}{a'}\sum_{l=2}^{\infty}\alpha^l\left(\frac{a'}{r'}\right)^{l+1}\left(\frac{r}{a}\right)^l P_l(\cos\psi) \tag{6.28}$$

where

$$\alpha = \frac{a}{a'} < 1 \tag{6.29}$$

is the ratio of the semimajor axes of the masses m and m'.

If we consider the terms with $l = 2$ then the series expansion for r/a given in §2.5 gives

$$\left(\frac{r}{a}\right)^2 \approx 1 - 2e\cos M + \left(\frac{1}{2}\right)e^2(3 - \cos 2M)$$

$$\left(\frac{a'}{r'}\right)^3 \approx 1 + 3e'\cos M' + \left(\frac{3}{2}\right)e'^2(1 + 3\cos 2M')$$

with

$$\begin{aligned}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 \approx &1 + \frac{3}{2}e^2 + \frac{3}{2}e'^2 - 2e\cos M + 3e'\cos M' \\ &- \frac{1}{2}e^2\cos 2M + \frac{9}{2}e'^2\cos 2M' \\ &- 3ee'\cos[M - M'] - 3ee'\cos[M + M'].\end{aligned}$$