ASTR430 Homework #5
Due Tuesday, April 17, 2001

1. Build a Planet!

a) Evaluate the moment of inertia I of a uniform density spherical planet of radius R about its spin axis by integrating the expression $dI = r^2 dm$ over the volume of the sphere (dI is the differential moment of inertia due to mass element dm at distance r from the spin axis). Write your answer two ways: i) in terms of R and ρ, the mass density and ii) in terms of R and M, the mass of the planet.

b) Will the moment of inertia increase or decrease for an oblate planet (one with an equatorial diameter greater than its polar diameter)? What about for a differentiated planet with a dense core and a less dense mantle? Explain your answers.

c) Use your answer from a) to get the moment of inertia of a two-layer planet with a core of radius R_c and density ρ_c, and a mantle of radius R and density ρ_m. Write your answer in terms of R, R_c, ρ_c, and ρ_m. Apply your result to get a constraint on the interior structure of Mars using the measured $I_{Mars} = 0.365MR^2$. Write the constraint in terms of R, R_c, ρ_c, ρ_m, and ρ_{AV}, where $\rho_{AV} = 3.93$ g/cm^3 is the average density of Mars.

d) Write down an expression for the total mass of the planet in terms of R, R_c, ρ_c, and ρ_m. Eliminate mass in favor of ρ_{AV} to get a second constraint on the interior structure of Mars.

e) Parts c) and d) give two constraints on the three unknowns R_c, ρ_c, and ρ_m. If we assume a core density for Mars, the system reduces to two equations in two unknowns. Eliminate the core radius from your two equations to get a single equation that relates the two unknown densities. Assume an iron core with density $\rho_c = 7.5$g/cm^3, and guess different ρ_m's until you find a solution (This equation cannot be solved analytically). What core radius R_c does your answer suggest?

2. Estimate the total amount of energy released when Helium rains out of Saturn's mantle into its metallic core. For the purpose of this problem, we will ignore ice and rock components and assume that Saturn is made of just Hydrogen (H) and Helium (He).

a) Start by obtaining the total masses of H and He in Saturn assuming solar abundances.

b) Next, write down a general integral for the potential energy of a spherically symmetric planet in which the planet's density ρ is a function of radius. Hint: consider successively adding thin shells of material to a spherical core by integrating over r, the radial coordinate (see the class handout).

c) Now assuming that Saturn's density is constant and that the planet is made of uniformly mixed H and He, calculate the planet's potential energy. You've just evaluated the integral for the case of a planet with uniform density. This is a test of your answer in part b); you should get $U = -3GM_S^2/(5R_S)$ as on the class handout.

d) Now calculate the potential energy of a fully differentiated Saturn assuming that it has a uniform density Helium core of radius R_c surrounded by a uniform Hydrogen mantle. The integral has several messy terms, but it could be evaluated; make sure that it reduces correctly in the limits $R_c = 0$ and $R_c = R_S$.

e) Assume that $R_c = 0.3R_S$ and evaluate your answer in part d). You can approximate. How much extra energy is available as heat when Saturn evolves from state c) to state d)? Is this a large or small amount of heat?