Lagrange's Planetary Equations

$$\frac{d\Omega}{dt} = \frac{nar}{\mu\sqrt{1-e^2}}\, N \sin u \operatorname{cosec} i,$$

$$\frac{di}{dt} = \frac{nar}{\mu\sqrt{1-e^2}}\, N \cos u,$$

$$\frac{de}{dt} = \frac{na^2}{\mu}\sqrt{1-e^2}\{R \sin v + B(\cos v + \cos E)\},$$

$$\frac{d\tilde{\omega}}{dt} = \frac{na^2}{\mu e}\sqrt{1-e^2}\left\{-R\cos v + B\left(1+\frac{r}{p}\right)\sin v\right\} + 2\sin^2\tfrac{1}{2}i\,\frac{d\Omega}{dt},$$

$$\frac{d\omega}{dt} = \frac{na^2}{\mu e}\sqrt{1-e^2}\left\{-R\cos v + B\left(1+\frac{r}{p}\right)\sin v\right\} - \cos i\,\frac{d\Omega}{dt},$$

$$\frac{da}{dt} = \frac{2na^2}{\mu}\left\{R\,\frac{ae}{\sqrt{1-e^2}}\sin v + B\,\frac{a^2\sqrt{1-e^2}}{r}\right\},$$

$$\frac{dn}{dt} = -\frac{3}{2}\frac{n}{a}\frac{da}{dt}.$$

Planetary Equation (Potential Form)

$$\frac{da}{dt} = 2\,\frac{na^2}{\mu}\,\frac{\partial \Re}{\partial \epsilon},$$

$$\frac{de}{dt} = \frac{na(1-e^2)}{\mu e}\,\frac{\partial \Re}{\partial \epsilon} - \frac{na\sqrt{1-e^2}}{\mu e}\left(\frac{\partial \Re}{\partial \epsilon} + \frac{\partial \Re}{\partial \tilde{\omega}}\right),$$

$$\frac{d\tilde{\omega}}{dt} = \frac{na\sqrt{1-e^2}}{\mu e}\,\frac{\partial \Re}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial \Re}{\partial i},$$

$$\frac{d\omega}{dt} = \frac{na\sqrt{1-e^2}}{\mu e}\,\frac{\partial \Re}{\partial e} - \frac{na}{\mu\sqrt{1-e^2}}\cot i\,\frac{\partial \Re}{\partial i},$$

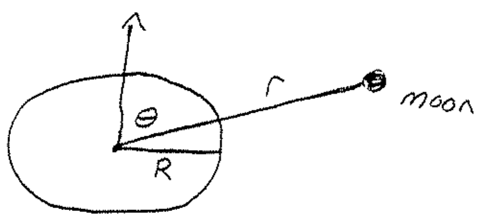
$$\frac{d\epsilon_1}{dt} = -\frac{2na^2}{\mu}\,\frac{\partial \Re}{\partial a} + \frac{na\sqrt{1-e^2}}{\mu e}\left(1-\sqrt{1-e^2}\right)\frac{\partial \Re}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial \Re}{\partial i},$$

$$\frac{d\Omega}{dt} = \frac{na}{\mu\sqrt{1-e^2}}\operatorname{cosec} i\,\frac{\partial \Re}{\partial i},$$

$$\frac{di}{dt} = -\frac{na}{\mu\sqrt{1-e^2}}\left\{\operatorname{cosec} i\,\frac{\partial \Re}{\partial \Omega} + \tan\tfrac{1}{2}i\left(\frac{\partial \Re}{\partial \epsilon} + \frac{\partial \Re}{\partial \tilde{\omega}}\right)\right\},$$

$$\frac{di}{dt} = -\frac{na}{\mu\sqrt{1-e^2}}\left\{\operatorname{cosec} i\,\frac{\partial \Re}{\partial \Omega} - \cot i\,\frac{\partial \Re}{\partial \omega}\right\}.$$

Orbits Around an Oblate Planet



Axisymmetric Potential

Potential: $V(r,\theta) = -\frac{GM}{r} - \frac{GM}{R} \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^{n+1} P_n(\theta)$

If (center of coordinates) = (center of mass)

$\Rightarrow J_1 = 0$

For Rotating Planets: $J_2 >> J_n$ w/ $n > 2$

so Perturbing Potential:

$$U_p(r,\theta) = -\frac{GM}{R} J_2 \left(\frac{R}{r}\right)^3 P_2(\theta)$$

$$V_p(r,\theta) = \frac{GmJ_2R^2}{2}\left(\frac{3\cos^2\theta - 1}{r^3}\right)$$

Express in terms of orbital elements

$$\cos\theta = \sin i \sin(\omega+\nu) \quad \text{(spherical trig)}$$

$$1/r^3 = \left(\frac{1+e\cos\nu}{a(1-e^2)}\right)^3$$

so $U_p(a,e,i,\Omega,\omega,\nu) = -\left(\frac{GmJ_2R^2}{2}\right) \frac{(1+e\cos\nu)^3(3\sin^2(i)\sin^2(\omega+\nu) - }{a^3(1-e^2)^3}$

$R = -U_p$ (celestial mechanics convention)

Average Up over one orbit

$$\langle R \rangle = \frac{1}{\tau}\int_0^{\tau} R\, dt \qquad \text{Kepler II: } \frac{dv}{dt} = h/r^2$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R r^2}{h}\, dv$$

Can show $\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3(1-e^2)^{3/2}}$

$$\left\langle \frac{\sin^2(w+v)}{r^3} \right\rangle = \frac{1}{2a^3(1-e^2)^{3/2}}$$

$$\Rightarrow \boxed{\langle R \rangle = \frac{GM J_2 R^2}{2a^3(1-e^2)^{3/2}}\left(\frac{3}{2}\sin^2 i - 1\right)}$$

$$\left\langle \frac{da}{dt} \right\rangle = \left\langle \frac{de}{dt} \right\rangle = \left\langle \frac{di}{dt} \right\rangle = 0$$

since $\langle R \rangle$ is not a function of Ω, w, or v.

$$\left\langle \frac{d\Omega}{dt} \right\rangle = \frac{-3 n J_2 R^2}{2a^2(1-e^2)^2}\cos i$$

line of nodes <u>regresses</u>

$$\left\langle \frac{dw}{dt} \right\rangle = \frac{3 n J_2 R^2}{2a^2(1-e^2)^2}\left(2 - \frac{5}{2}\sin^2 i\right)$$

argument of pericenter <u>precesses</u> for $i < 63°$

$$\left\langle \frac{dm}{dt} \right\rangle = n + \frac{3 n J_2 R^2}{2a^2(1-e^2)^{3/2}}\left(1 - \frac{3}{2}\sin^2 i\right)$$

moon orbits faster than Kepler's 3rd law

Orbit-Averaged Equations for an oblate Planet

$$\left\langle \frac{da}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{de}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{di}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{d\Omega}{dt} \right\rangle_{J_2} = -\frac{3nJ_2R_{\mathrm{p}}^2}{2a^2(1-e^2)^2}\cos i,$$

$$\left\langle \frac{d\omega}{dt} \right\rangle_{J_2} = \frac{3nJ_2R_{\mathrm{p}}^2}{2a^2(1-e^2)^2}\left(2-\frac{5}{2}\sin^2 i\right),$$

$$\left\langle \frac{d\mathcal{M}}{dt} - n \right\rangle_{J_2} = \frac{3nJ_2R_{\mathrm{p}}^2}{2a^2(1-e^2)^{3/2}}\left(1-\frac{3}{2}\sin^2 i\right),$$