ASTR450 Homework #12 – The Last HW!
Due Thursday, December 7

1. Projectiles (a,b: Easy, c,d: Moderate, e: Hard). Consider a classic Freshman physics problem: projectiles launched at low velocity in a constant gravity field, g.
a) As a warm-up, use conservation of Energy to solve for the maximum height H reached by the projectile launched at speed v and at an angle θ with the horizontal assuming constant g.
b) Subject you solution from part a) to various limits to test its validity. Then determine whether there is an escape velocity for this problem.
c) You launch a rocket straight up ($\theta = 90°$) from the Earth's North pole and it rises up then falls back to Earth. Accounting for true $1/r^2$ gravity, the maximum height above Earth's surface H is given by one of the expressions below. Here R_E is the Earth's radius, $X = v^2 R_E/GM_E$, G is the gravitational constant, M_E is the Earth's mass and v is the launch velocity. Rule out as many of the following expressions as you can using units, limits, and symmetries.

A) $H = R_E X/(1 + \sqrt{X})$
B) $H = R_E X/(1 - X)$
C) $H = R_E X/(2 - X)$
D) $H = R_E(1 - X)/(2 - X)$
E) $H = vX^2/(2 - X)$
F) $H = R_E X/2$
G) $H = R_E X^2/(2 - X)$
H) $H = R_E X|1 - X|/(2 - X)$

d) Now derive an expression for H for a vertically-launched projectile accounting for $1/r^2$ gravity. Treat the Earth as a perfect sphere and neglect its spin. Check your answer against part c).
e) Finally, extend your solution for the maximum height of a projectile in part d) to an arbitrary launch angle θ and check your answer against part d). Start by arguing that trajectories will be segments of conic sections. What conservation laws can you use? Your answer will be algebraically messy - simplify it as much as you can by putting your final answer in terms of the dimensionless parameter from part c). Does your answer make sense? Consider all of the limiting cases that you can think of as tests – including your answers from c) and d). Write down your tests and use them to argue that your full answer is reasonable (or unreasonable!).

2. (Moderate) Pluto and Neptune are linked by a 3:2 eccentricity-type resonance. The resonant term in the disturbing function felt by Pluto (due to Neptune) is $\mathcal{R}_N = M_N \beta e_P \cos(3\lambda_P - 2\lambda_N - \varpi_P)$ while the term felt by Neptune (due to Pluto) is $\mathcal{R}_P = M_P \beta e_P \cos(3\lambda_P - 2\lambda_N - \varpi_P)$. Here "P" subscripts refer to Pluto, "N" subscripts refer to Neptune, M, e, λ, and ϖ are mass, eccentricity, mean longitude, and longitude of pericenter, and β is a constant.
a) List all other first-order 3:2 resonant terms for Neptune's perturbations on Pluto. List all other second-order resonant terms.
b) Find the time rates of change of the orbital elements $(a, e, i, \Omega, \varpi)$ of Pluto and Neptune due to disturbing functions $\mathcal{R}_N$ and $\mathcal{R}_P$. Remember that $d\mathcal{R}/d\epsilon = d\mathcal{R}/d\lambda$ and $p = a(1-e^2)$. Give your answer to lowest order in eccentricity and inclination, and keep only the most perturbed element in each of these sets: $(de_N/dt, di_N/dt, de_P/dt, di_P/dt)$ and $(d\Omega_N/dt, d\varpi_N/dt, d\Omega_P/dt, d\varpi_P/dt)$. You can set the other members of each set equal to zero.
c) Show that the sum of the orbital energy of Pluto plus that of Neptune is conserved to lowest order in e, i.