(18)

Resonances

Orbital Resonances

Jupiter - Asteroid Belt
Neptune - Kuiper Belt
Saturn moons - Saturn Ring
Jupiter - Trojan Asteroids
Mars - Trojan Asteroid
Io - Europa - Ganymede
Dione - Enceladus - Helene
Tethys - Mimas - Telesto - Calypso
Titan - Hyperion
Others in the past?
Neptune - Pluto

Solar B-field - Zodiacal Dust
Jupiter's Ring
Saturn's E Ring
Neptune Dust ?
Dust from Phobos

SPIN Resonances

Mercury
Venus
Moon + all tidally locked satellites
Hyperion & Nereid

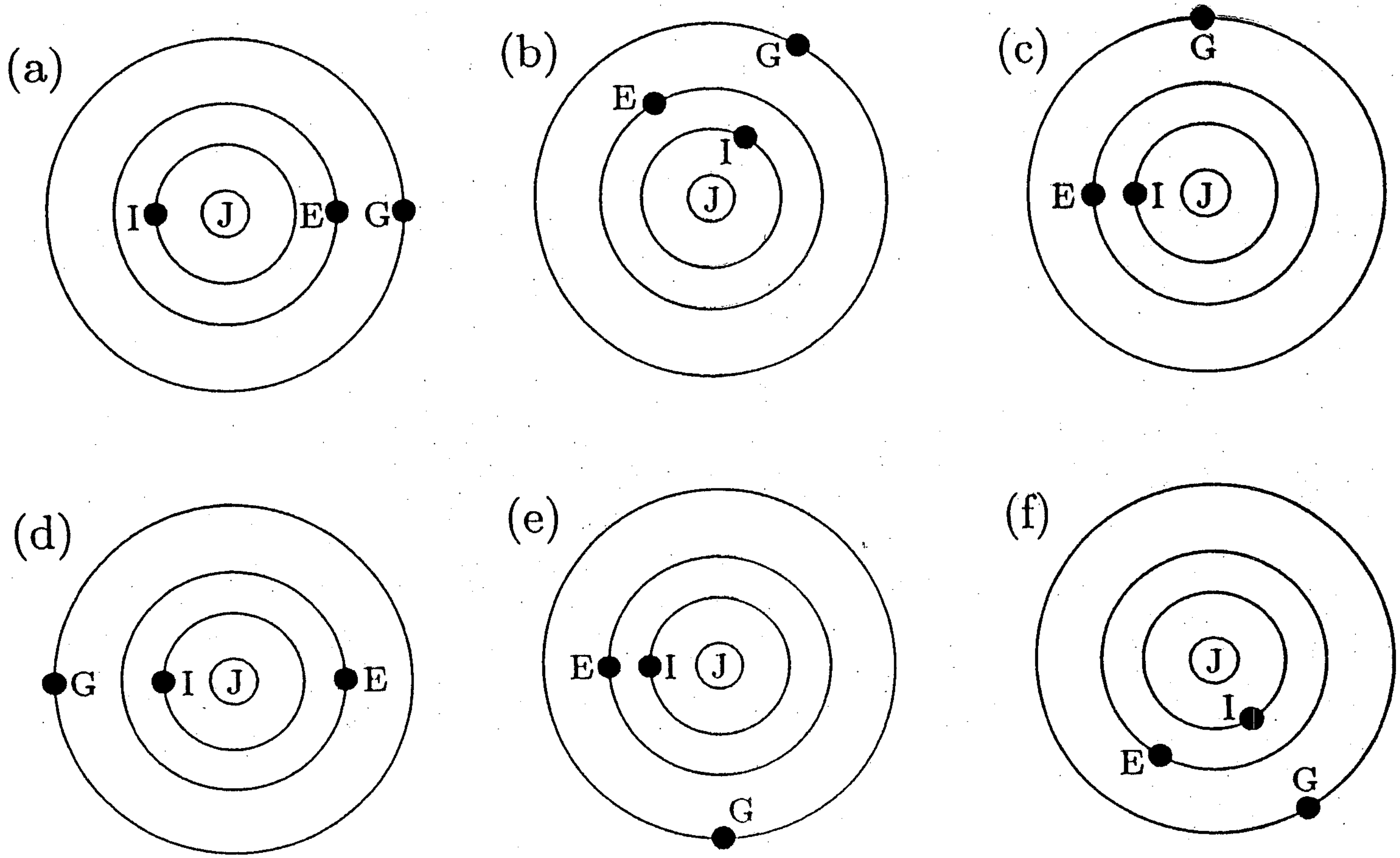


Figure 8.24 The sequence of conjunctions for the Galilean satellites. The configurations at times (a) $t = 0$, (b) $t = T_{\text{rep}}/6$, (c) $t = T_{\text{rep}}/4$, (d) $t = T_{\text{rep}}/2$, (e) $t = 3T_{\text{rep}}/4$, and (f) $t = 5T_{\text{rep}}/6$. The letters J, I, E and G denote Jupiter, Io, Europa and Ganymede respectively.

Orbital Resonances

One dimensional analog

$$\ddot{x} + \omega_0^2 x = f \cos \omega t$$

if $\omega \sim \omega_0$ $\Rightarrow$ Resonant Forcing (large effects)

(forcing frequency) (natural frequency)

Orbits around planets

6 dimensions $(a, e, i, \Omega, \tilde{\omega}, \varepsilon)$

$\left\{ \text{forcing frequencies} \right\}$ $\left\{ \text{natural frequencies} \right\}$

Perturbation Theory

- perturbing force is small compared to direct gravity.

⇒ orbital elements change slowly in time

- Arbitrary Perturbation = Secular Perturbations + Resonant Perturbations

Secular Perturbations ↑ do not depend on perturbee's longitude

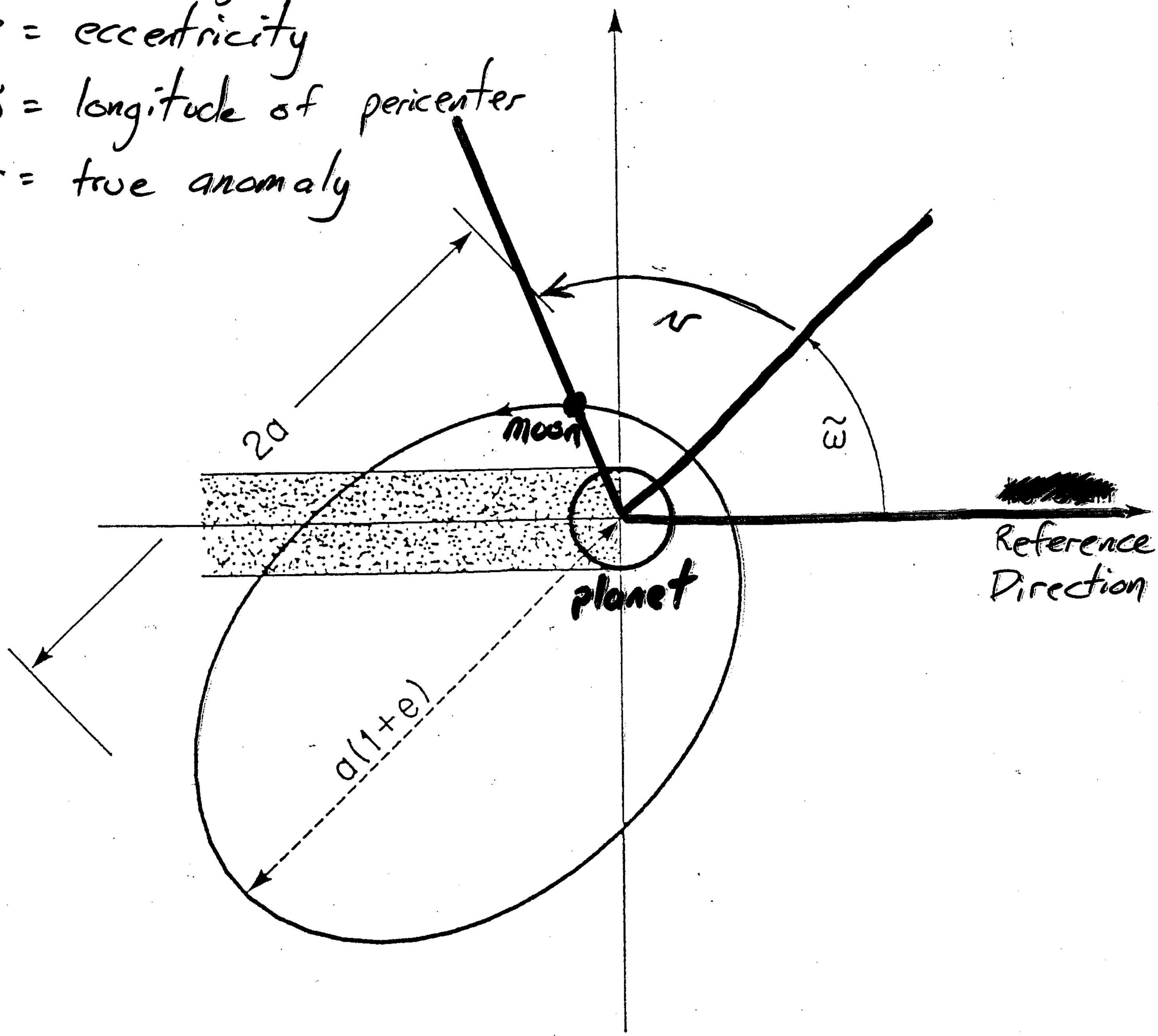
Resonant Perturbations ↑ do depend on perturbee's longitude

Secular Perturbations:
- tidal evolution of the moon
- planetary perturbations
- oblate planet

Resonant Perturbations
- some moon-moon interactions

Orbital Elements

a = semimajor axis
e = eccentricity
$\tilde{\omega}$ = longitude of pericenter
ν = true anomaly



4 orbital elements define a planar orbit. 2 more define

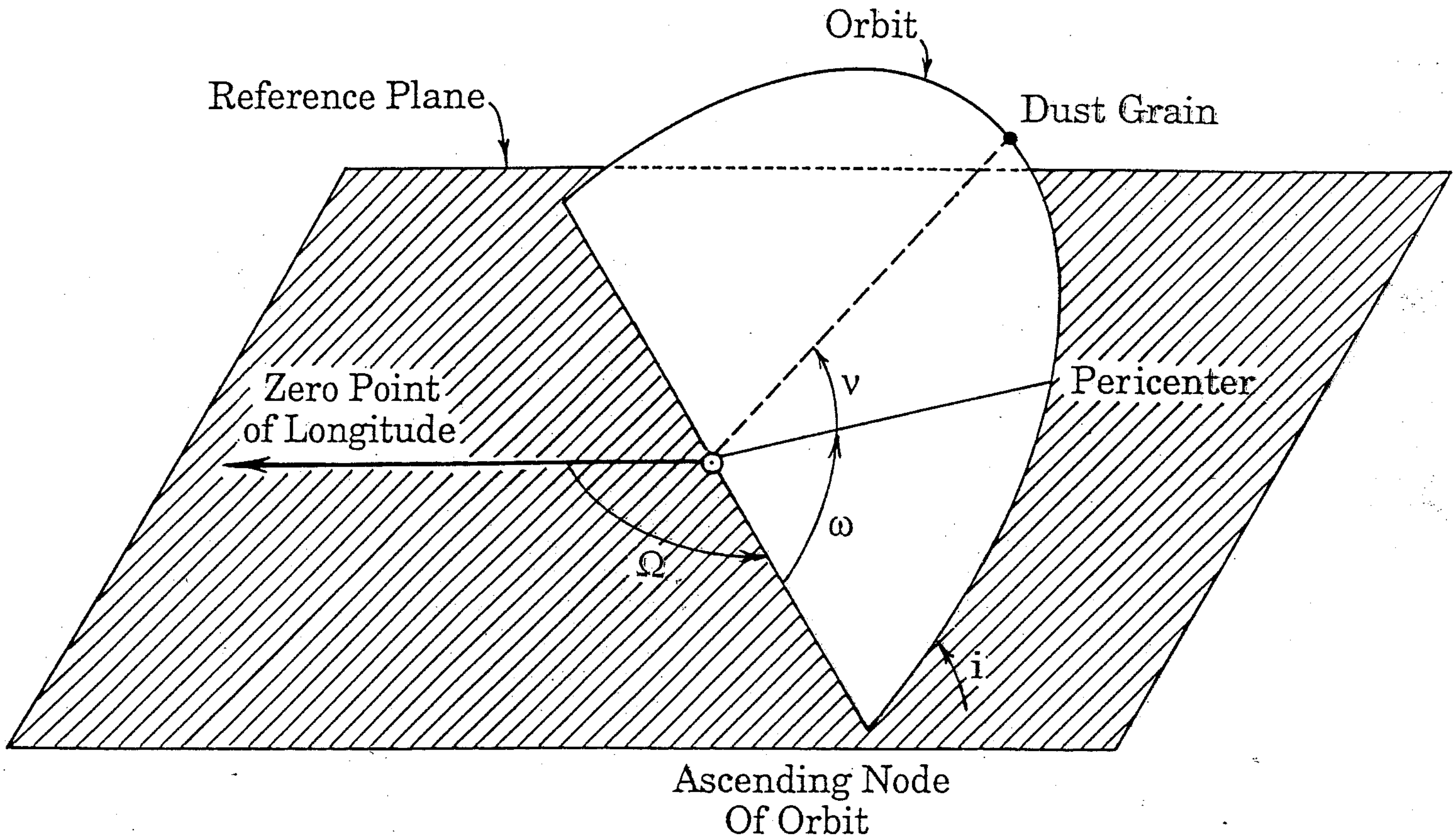

Orbit
Reference Plane
Dust Grain
Zero Point
of Longitude
ν
Pericenter
ω
Ω
i
Ascending Node
Of Orbit

Fig 1b

Orbital Elements

1. a — semimajor axis
   q — $=a(1-e)$ pericenter distance

2. e — eccentricity

3. i — inclination

4. Ω — longitude of the ascending node

5. $\tilde{\omega}$ — longitude of pericenter
   ω — argument of pericenter

6. ν — true anomaly
   E — eccentric anomaly
   M — mean anomaly
   u — argument of latitude
   ℓ — true longitude
   T — time of pericenter passage

Common Sets:

$q, e, i, \Omega, \omega, \nu$

$a, e, i, \Omega, \tilde{\omega}, \ell$

$q, e, i, \Omega, \omega, T$

Lagrange's Planetary Equations

$$\frac{d\Omega}{dt} = \frac{nar}{\mu\sqrt{1 - e^2}} N \sin u \operatorname{cosec} i,$$

$$\frac{di}{dt} = \frac{nar}{\mu\sqrt{1 - e^2}} N \cos u,$$

$$\frac{de}{dt} = \frac{na^2}{\mu}\sqrt{1 - e^2}\{R \sin v + B(\cos v + \cos E)\},$$

$$\frac{d\tilde{\omega}}{dt} = \frac{na^2}{\mu e}\sqrt{1 - e^2}\left\{-R \cos v + B\left(1 + \frac{r}{p}\right) \sin v\right\} + 2 \sin^2 \tfrac{1}{2}i \frac{d\Omega}{dt},$$

$$\frac{d\omega}{dt} = \frac{na^2}{\mu e}\sqrt{1 - e^2}\left\{-R \cos v + B\left(1 + \frac{r}{p}\right) \sin v\right\} - \cos i \frac{d\Omega}{dt},$$

$$\frac{da}{dt} = \frac{2na^2}{\mu}\left\{R \frac{ae}{\sqrt{1 - e^2}} \sin v + B \frac{a^2\sqrt{1 - e^2}}{r}\right\},$$

$$\frac{dn}{dt} = -\frac{3}{2}\frac{n}{a}\frac{da}{dt}.$$

Planetary Equations (Potential Form)

$$\frac{da}{dt} = 2\,\frac{na^2}{\mu}\frac{\partial \mathfrak{R}}{\partial \epsilon},$$

$$\frac{de}{dt} = \frac{na(1-e^2)}{\mu e}\frac{\partial \mathfrak{R}}{\partial \epsilon} - \frac{na\sqrt{1-e^2}}{\mu e}\left(\frac{\partial \mathfrak{R}}{\partial \epsilon} + \frac{\partial \mathfrak{R}}{\partial \tilde{\omega}}\right),$$

$$\frac{d\tilde{\omega}}{dt} = \frac{na\sqrt{1-e^2}}{\mu e}\frac{\partial \mathfrak{R}}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial \mathfrak{R}}{\partial i},$$

$$\frac{d\omega}{dt} = \frac{na\sqrt{1-e^2}}{\mu e}\frac{\partial \mathfrak{R}}{\partial e} - \frac{na}{\mu\sqrt{1-e^2}}\cot i\,\frac{\partial \mathfrak{R}}{\partial i},$$

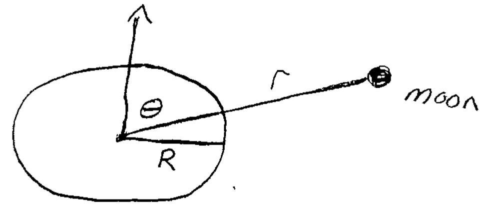
$$\frac{d\epsilon_1}{dt} = -\frac{2na^2}{\mu}\frac{\partial \mathfrak{R}}{\partial a} + \frac{na\sqrt{1-e^2}}{\mu e}\left(1-\sqrt{1-e^2}\right)\frac{\partial \mathfrak{R}}{\partial e} + \frac{na}{\mu\sqrt{1-e^2}}\tan\tfrac{1}{2}i\,\frac{\partial \mathfrak{R}}{\partial i},$$

$$\frac{d\Omega}{dt} = \frac{na}{\mu\sqrt{1-e^2}}\operatorname{cosec} i\,\frac{\partial \mathfrak{R}}{\partial i},$$

$$\frac{di}{dt} = -\frac{na}{\mu\sqrt{1-e^2}}\left\{\operatorname{cosec} i\,\frac{\partial \mathfrak{R}}{\partial \Omega} + \tan\tfrac{1}{2}i\left(\frac{\partial \mathfrak{R}}{\partial \epsilon} + \frac{\partial \mathfrak{R}}{\partial \tilde{\omega}}\right)\right\},$$

$$\frac{di}{dt} = -\frac{na}{\mu\sqrt{1-e^2}}\left\{\operatorname{cosec} i\,\frac{\partial \mathfrak{R}}{\partial \Omega} - \cot i\,\frac{\partial \mathfrak{R}}{\partial \omega}\right\}.$$

Orbits Around an Oblate Planet



Axisymmetric Potential

Potential: $V(r,\theta) = -\frac{GM}{r} - \frac{GM}{R}\sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^{n+1} P_n(\theta)$

If (center of coordinates) = (center of mass)

$\Rightarrow J_1 = 0$

For Rotating Planets: $J_2 \gg J_n$ w/ $n>2$

so Perturbing Potential:

$$U_p(r,\theta) = -\frac{GM}{R} J_2 \left(\frac{R}{r}\right)^3 P_2(\theta)$$

$$V_p(r,\theta) = \frac{GMJ_2R^2}{2}\left(\frac{3\cos^2\theta - 1}{r^3}\right)$$

Express in terms of orbital elements

$$\cos\theta = \sin i \sin(\omega+\nu) \qquad \text{(spherical trig)}$$

$$1/r^3 = \left(\frac{1+e\cos\nu}{a(1-e^2)}\right)^3$$

so $U_p(a,e,i,\Omega,\omega,\nu) = -\left(\frac{GMJ_2R^2}{2}\right)\frac{(1+e\cos\nu)^3(3\sin^2(i)\sin^2(\omega+\nu)-1)}{a^3(1-e^2)^3}$

$R = -U_p$ (celestial mechanics convention)

Average U_p over one orbit

$$\langle R \rangle = \frac{1}{T}\int_0^T R\,dt \qquad \text{Kepler II: } \frac{dv}{dt} = h/r^2$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R r^2}{h}\,dv$$

Can show $\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3(1-e^2)^{3/2}}$

$$\left\langle \frac{\sin^2(\omega+v)}{r^3} \right\rangle = \frac{1}{2a^3(1-e^2)^{3/2}}$$

$$\Rightarrow \boxed{\langle R \rangle = \frac{GmJ_2R^2}{2a^3(1-e^2)^{3/2}}\left(\frac{3}{2}\sin^2 i - 1\right)}$$

$$\left\langle \frac{da}{dt} \right\rangle = \left\langle \frac{de}{dt} \right\rangle = \left\langle \frac{di}{dt} \right\rangle = 0$$ since $\langle R \rangle$ is not a function of Ω, ω, or v.

$$\left\langle \frac{d\Omega}{dt} \right\rangle = \frac{-3nJ_2R^2}{2a^2(1-e^2)^2}\cos i$$ line of nodes <u>regresses</u>

$$\left\langle \frac{d\omega}{dt} \right\rangle = \frac{3nJ_2R^2}{2a^2(1-e^2)^2}\left(2 - \frac{5}{2}\sin^2 i\right)$$ argument of pericenter <u>precesses</u> for $i < 63°$

$$\left\langle \frac{dm}{dt} \right\rangle = n + \frac{3nJ_2R^2}{2a^2(1-e^2)^{3/2}}\left(1 - \frac{3}{2}\sin^2 i\right)$$ moon orbits faster than Kepler's 3rd law

Orbit-Averaged Equations for an oblate Planet

$$\left\langle \frac{da}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{de}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{di}{dt} \right\rangle_{J_2} = 0,$$

$$\left\langle \frac{d\Omega}{dt} \right\rangle_{J_2} = -\frac{3nJ_2R_\mathrm{p}^2}{2a^2(1-e^2)^2}\cos i,$$

$$\left\langle \frac{d\omega}{dt} \right\rangle_{J_2} = \frac{3nJ_2R_\mathrm{p}^2}{2a^2(1-e^2)^2}\left(2-\frac{5}{2}\sin^2 i\right),$$

$$\left\langle \frac{d\mathcal{M}}{dt} - n \right\rangle_{J_2} = \frac{3nJ_2R_\mathrm{p}^2}{2a^2(1-e^2)^{3/2}}\left(1-\frac{3}{2}\sin^2 i\right),$$

Planetary Perturbations

1. Obtain the perturbing potential

2. Translate to orbital elements

3. Taylor expand in small quantities
   $(e, e', i, i', a'/a)$

4. Combine trig functions to obtain a sum over terms of the form
   $$f(a'/a, e, e', i, i') \cos(A\lambda + B\lambda' + C\tilde{\omega} + D\tilde{\omega}' + E\Omega + F\Omega')$$

5. Take derivatives to obtain the time-rates of change of the orbital elements