

ASTR450 Homework # 9 – Perturbations  
Due Thursday, April 23

1. **Impulse Approximation for Atmospheric Drag.** Consider an artificial satellite on an inclined elliptical orbit acted on by a drag force of the form  $\mathbf{F} = -k\mathbf{v}_{rel}$ , where  $\mathbf{v}_{rel}$  is the velocity of the satellite relative to the atmosphere, and  $k$  is a positive constant. Recall that Earth's atmosphere decays exponentially with height (scale height  $\approx 10\text{km}$ ).

a) Approximate the range of eccentricities for which the drag force can be approximated by an impulse at pericenter.

b) Equatorial Orbits ( $i = \Omega = 0$ ). Consider a non-rotating Earth first. Describe how the orbital elements  $a, e, i, \Omega, \varpi$  vary in time. Show that the pericenter distance  $a(1-e)$  is unchanged by the perturbation. Why must this be so? Now describe, quantitatively, how rotation of an assumed spherical Earth affects your answer.

c) Finally imagine orbits with  $i \neq 0, \Omega \neq 0$  around a rotating Earth. Using the perturbation equations and other physical arguments, describe qualitatively how these orbits will evolve under atmospheric drag (i.e. how  $a, e, i, \Omega, \varpi$  vary in time).

2. **Radial Perturbation Forces.** Consider a radial perturbation force of the form  $\mathbf{F} = R\hat{r}$ , where  $R$  is a function of the distance  $r$ .

a) Apply the perturbation equations to this force and obtain simplified expressions for  $da/dt$ ,  $de/dt$ ,  $di/dt$ ,  $d\Omega/dt$ , and  $d\varpi/dt$ .

b) A radial perturbation to gravity, which is itself a radial force, is an example of a central force. So angular momentum must be conserved. Show that your equations conserve the angular momentum vector and describe the constraints that this imposes on these orbits.

c) Now let  $R = Ar^n$  where  $A$  is a constant. Take the time average of your expressions over a single unperturbed Keplerian orbit (this step assumes that the perturbation is small). Show that  $\langle r^n \sin \nu \rangle = 0$  and argue, on physical ground, that  $\langle \cos \nu \rangle$  is negative (or zero) and that  $\langle r^{-2} \cos \nu \rangle = 0$ . It can be shown that  $\langle r^n \cos \nu \rangle$  is negative for  $n > -2$  and positive for  $n < -2$ . Use this fact to determine how the sign of your time-averaged  $d\varpi/dt$  depends on  $A$  and  $n$ . Use the Central Force Integrator to check your results numerically.

d) Finally, consider the General Relativistic (GR) Perturbation  $R = Ar^{-4}$  where  $A$  is a small negative constant. The integral  $\langle r^{-4} \cos \nu \rangle = a^{-4}e(1-e^2)^{-5/2}$ . Solve, analytically, for the value of  $A$  that will give 30 degrees of precession per orbit for  $e = 0.5$  (The true effects of GR on Mercury's orbit are almost exactly a million times weaker). Convert your prediction into the proper initial conditions for the Central Force Integrator, and test it! Start your orbit at pericenter and turn in a copy of your plot.