

## 1. Convection in Stars

While it is possible to construct a solution to the equations of structure using only the radiative equation for  $dT/dr$  – the equation with the opacity  $\kappa$  – such a solution would in most cases contain regions that are unstable against convection. Because convective motions move thermal energy from one layer to another, convection will change the temperature gradient in a star and invalidate a radiative-only solution.

We can obtain the criterion for stability by considering the displacement of a blob of gas from a point where the density and pressure are  $(\rho_1, P_1)$  upwards to a point where the surrounding conditions are  $(\rho_2, P_2)$ . After this displacement, the density and pressure *inside* the blob will be  $(\rho_*, P_*)$ . Now pressure differences will equalize at the speed of sound, so we can assume that  $P_* = P_2$ . If the blobs are large, the heat flow between the blob and surrounding material will be slow – in fact, we can neglect heat flow and consider the changes inside the blob to be *adiabatic*. Then all we need do is compare the density inside the displaced blob  $\rho_*$ , with that outside  $(\rho_2)$  as shown in the figure below.

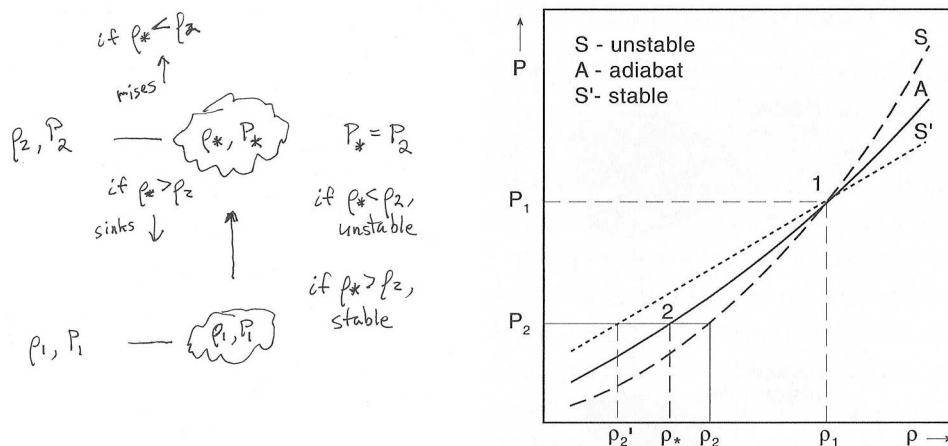


Fig. 1.— Schematic density-pressure diagram illustrating the criterion for stability against convection.

We see from the graph that if the slope of the curve,  $dP/d\rho$ , is like S, i.e., steeper than the adiabatic curve A, then  $\rho_* < \rho_2$ , the blob will continue to rise, and the gas is unstable against convection. Conversely, if the slope is shallower (like S')  $\rho_* > \rho_2$ , the blob will sink back and the gas is stable. So we see we can conclude that there will be *stability against convection* as long as

$$\left(\frac{dP}{d\rho}\right)_{star} < \left(\frac{dP}{d\rho}\right)_{adiabatic} \tag{1}$$

Let us recall the result we obtained earlier for the case of an adiabatic change:

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

Now the specific volume  $V = 1/\rho$ , so we have

$$\frac{dV}{dx} = -\frac{1}{\rho^2} \frac{d\rho}{dx} = -V \frac{1}{\rho} \frac{d\rho}{dx} \implies \frac{dV}{V} = -\frac{d\rho}{\rho}$$

and thus, for an adiabatic change,

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \implies \left( \frac{\rho}{P} \frac{dP}{d\rho} \right)_{adiabatic} = \gamma \quad (2)$$

Now since  $\rho/P$  is a positive number, we can multiply the condition above [eq. (1)] by that factor to obtain

$$\frac{\rho}{P} \left( \frac{dP}{d\rho} \right)_{star} < \frac{\rho}{P} \left( \frac{dP}{d\rho} \right)_{ad} = \gamma \quad (3)$$

This is fine, but a stability criterion in terms of the temperature gradient is actually more useful. Let's start with the ideal gas law, and with its differential form:

$$P = \frac{N_A k}{\mu} \rho T \quad \text{and} \quad dP = \frac{N_A k}{\mu} \cdot d\rho \cdot T + \frac{N_A k}{\mu} \cdot \rho \cdot dT$$

Divide the second equation by the first:

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

and multiply through by  $P/dP$  :

$$1 = \frac{P}{\rho} \frac{d\rho}{dP} + \frac{P}{T} \frac{dT}{dP} \implies \frac{\rho}{P} \frac{dP}{d\rho} = \left[ 1 - \left( \frac{P}{T} \frac{dT}{dP} \right) \right]^{-1} \quad (4)$$

This is a general relation for an ideal gas and applies to the gradients in the star as well as the adiabatic changes in the moving blob of gas.

Now look at the stability criterion [eq (3)]. We can replace both sides by the relation we've just obtained to get

$$\left[ 1 - \left( \frac{P}{T} \frac{dT}{dP} \right)_{star} \right]^{-1} < \left[ 1 - \left( \frac{P}{T} \frac{dT}{dP} \right)_{ad} \right]^{-1}$$

Both sides of inequality (3) are positive since the pressure must increase with density. Let  $a$  represent the logarithmic derivative on the left of the inequality and  $b$  that on the right. Then we can simplify the expression above as follows:

$$\left( \frac{1}{1-a} \right) < \left( \frac{1}{1-b} \right) \implies (1-a) > (1-b) \implies (-a) > (-b) \implies a < b$$

We therefore arrive at the criterion for stability:

$$\left( \frac{P}{T} \frac{dT}{dP} \right)_{star} < \left( \frac{P}{T} \frac{dT}{dP} \right)_{ad}$$

From equations (3) and (4), we see that logarithmic derivative for adiabatic changes is just

$$\left(\frac{P}{T} \frac{dT}{dP}\right)_{ad} = \frac{\gamma - 1}{\gamma} \equiv \nabla_{ad},$$

where it is customary to use the symbol  $\nabla$  (“nabla”) to represent the logarithmic derivative of temperature with respect to pressure. For an ideal monatomic gas,  $\gamma = 5/3$ , so  $\nabla_{ad} = 2/5$ .

So we can write the criterion for stability as

$$\left(\frac{P}{T} \frac{dT}{dP}\right)_{star} < \nabla_{ad} \quad \text{Schwarzschild's criterion for stability.}$$

If the logarithmic derivative of the temperature with respect to the pressure exceeds 0.4, convection will commence. Actually, it should be pointed out that there are many circumstances in stellar interiors where the value of  $\gamma < 5/3$  and hence  $\nabla_{ad} < 0.4$ . Examples are (1) when radiation pressure is important, and (2) when a major constituent of the gas is partially ionized. This makes convection more likely.

Let us return to our equation for the temperature gradient in a star when the energy flow is by radiation:

$$\frac{dT}{dr} = - \frac{3L_r \kappa \rho}{4\pi r^2 4ac T^3}$$

Now we know what to do: We integrate this equation, but at each point we check to see if the Schwarzschild criterion is violated. If it should be, we observe that the transport of energy by convection is so effective, that only the smallest difference from the adiabatic gradient is sufficient to carry the luminosity. Therefore, we can drop the previous equation and switch to one describing a purely adiabatic gradient:

$$\frac{dT}{dr} = \nabla_{ad} \frac{T}{P} \frac{dP}{dr}$$

References: Maoz, pp 58-61; Choudhuri, pp 67-69