1. Degenerate Pressure

We next consider a "Fermion gas" in quite a different context: the interior of a white dwarf star. Like other stars, white dwarfs have fully ionized plasma interiors. The positively charged particles consist of nuclei of He, C, and O, depending on the composition (which depends on the progenitor star). The negatively charged particles are electrons.

Unlike ordinary main sequence stars, white dwarfs are so dense that quantum effects are important – essentially because the electrons are packed so tightly together that they become a degenerate gas – i.e., one in which the momenta and energies are dictated by the "size" of the quantum states. The physical effect that arises when many electrons are confined in a small volume is called electron degeneracy pressure. This results from the random, large momenta that electrons must adopt because they cannot share the same quantum state.

You may ask why it is only the electrons which exert this pressure. This is because, at a given temperature, there are far fewer momentum states available to electrons than to ions. Consider the non-relativistic case, where $p = mv$ and $E = \frac{1}{2}mv^2$, so that $v = \sqrt{2E/m}$. As a result, $p = mv = \sqrt{2Em}$. Thus, at a given temperature, while the electrons and ions have the same average energy, the ions have far more momentum. The number of states in momentum space goes as p^3 , and thus as $m^{3/2}$. Consider a carbon nucleus with a mass 22030 times that of the electron: The carbon ion has access to 3.27×10^6 more momentum states than does the electron.

Unlike the case for the early universe when particles are freely created and destroyed, there is a fixed number of electrons inside a white dwarf, so we must include the chemical potential when we write down the occupation number (see our earlier notes). So number of electrons per unit volume of momentum space is:

$$n(p) \;=\; \frac{8\pi}{h^3} \;< N(E) > \;=\; \frac{8\pi}{h^3}\,\frac{1}{e^{(E-\mu)/kT} \;+\; 1} \tag{1}$$

In fact, white dwarfs are so dense that μ/kT is very large, and essentially all states with energy less than the Fermi energy $E_f = \mu$ are filled. We will return to this anon.

1.1. Relations from Special Relativity

Since the degenerate electrons span the range from non-relativistic to highly relativistic, we need to use some basic results from special relativity. First, the total energy of a particle is $E = (p^2c^2 + m^2c^4)^{1/2}$. This includes the rest-mass energy mc^2, so if we want only the kinetic energy $\mathcal{E}(p)$, it is given by

$$\mathcal{E}(p) \;=\; E - mc^2 \;=\; mc^2\,\left\{\sqrt{1 + (p/mc)^2} \;-\; 1\right\} \quad , \tag{2}$$

with the limiting values $\mathcal{E}(p) \rightarrow mc^2[\frac{1}{2}(p/mc)^2] \rightarrow p^2/2m$ for $p << mc$ and $\mathcal{E}(p) \rightarrow pc$ for $p >> mc$.

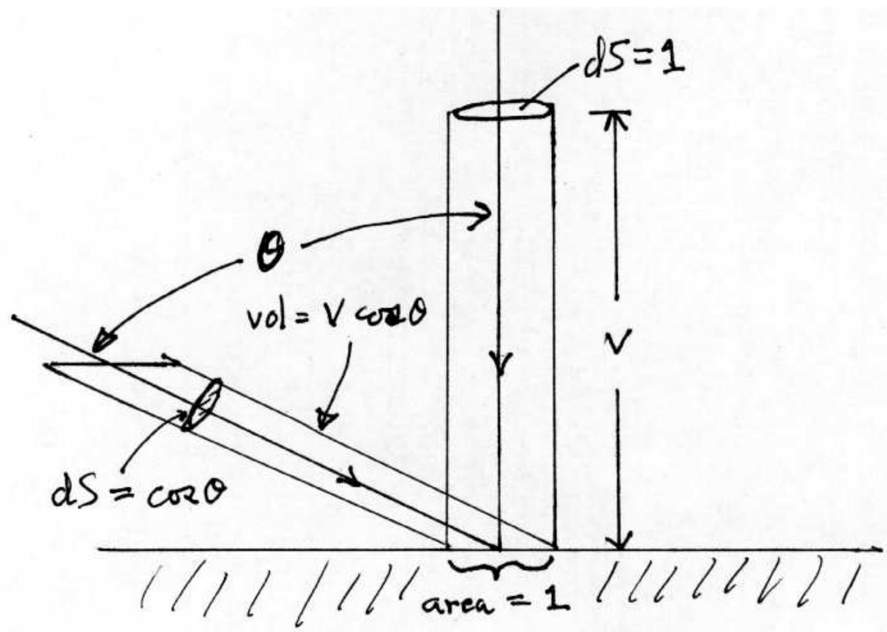
We also need the velocity of our particle as a function of p, which is

$$v = \frac{\partial \mathcal{E}}{\partial p} = \frac{pc^2}{(p^2c^2 + m^2c^4)^{1/2}} \,, \qquad (3)$$

with limiting values $v \to p/m$ for $p << mc$ and, of course, $v \to c$ for $p >> mc$.

1.2. The Pressure of the Electron Gas

To compute the pressure, we need to consider the rate at which momentum is transferred to a wall by the particles. Consider a cylinder which makes an angle θ with the perpendicular to the wall. Let the area of the wall cutting the cylinder be unity. Then, because of its inclination, the cross-sectional area of this cylinder will be $dS = \cos\theta$. Let the length of the cylinder be v, the velocity of the particle. Then, in one unit of time, all the particles with this momentum vector $\vec{p}$ in the cylinder can reach the wall. The volume of this cylinder is $vol = v\cos\theta$. Now the momentum of the particle perpendicular to the wall is $p_\perp = |\vec{p}|\cos\theta$, and reflection from the wall will reverse the direction of $p_\perp$ so that the total momentum transferred must be $2p\cos\theta$.



Taking all these factors into consideration, to get the pressure we must multiply the number of particles per unit physical space and per unit momentum space by the factor $2v\ p\ \cos^2\theta$ and integrate over the hemisphere bounded by the wall. Since the momentum distribution is isotropic, we can carry out the integration over angles directly:

$$\frac{1}{4\pi}\int_{hemisphere} 2\cos^2\theta \, d\Omega = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos^2\theta \, \sin\theta \, d\theta = \int_0^{\pi/2} \cos^2\theta \, \sin\theta \, d\theta \qquad (4)$$

The usual change of variable is $\mu = \cos\theta$ and $d\mu = -\sin\theta \; d\theta$, so that

$$\int_0^{\pi/2} \cos^2\theta \;\sin\theta \; d\theta \;=\; \int_1^0 \mu^2 \;(-d\mu) \;=\; \int_0^1 \mu^2 \; d\mu \;=\; \frac{1}{3} \tag{5}$$

So we obtain a very general equation for the pressure as the integral over the momentum p:

$$\boxed{P \;=\; \frac{1}{3}\int_0^\infty n(p)\; p \; v(p)\; 4\pi p^2 \; dp \qquad (6)}$$

Since there are two spin states, two electrons can occupy each cell in momentum space. Thus we have for our Fermion gas

$$n(p) \;=\; \frac{2}{h^3}\,\frac{1}{e^{(E-\mu)/kT}+1} \tag{7}$$

Looking back at $v(p)$ (equation 3), we see that the integral for P will be quite a mess. And we don't know what μ is either! Actually, we should determine it as a function of the number density of electrons, n, and the temperature T. We would do this by writing the expression for n as the integral of $n(p)$:

$$n \;=\; \int_0^\infty n(p) 4\pi p^2 \; dp \;=\; \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 \; dp}{e^{(E-\mu)/kT}+1} \tag{8}$$

If we can evaluate this we can obtain μ. It turns out that if the gas is non-degenerate, then $(-\mu/kT) >> 1$ so that we can ignore the +1 in the denominator. It is then easy to integrate equation (8), and a few steps gives us the ideal gas law. But we are interested in highly degenerate conditions. We approach this by writing the pressure as

$$P \;=\; \frac{8\pi}{3h^3}\int_0^\infty F(\mathcal{E})\; v(p)\; p^3 \; dp \;\;, \tag{9}$$

where

$$F(\mathcal{E}) \;=\; \frac{1}{\exp[(\mathcal{E}-(\mu - mc^2))/kT]+1} \;=\; \frac{1}{\exp[(\mathcal{E}-\mathcal{E}_F)/kT]+1} \;\;, \tag{10}$$

and the quantity $\mathcal{E}_F = (\mu - mc^2)$ is known as the *Fermi energy*. Under degenerate conditions, kT will be small compared to $\mathcal{E}$ – complete degeneracy is the limit as $T \to 0$. Let us look at the behavior of $F(\mathcal{E})$ as $\mathcal{E}$ ranges from less than $\mathcal{E}_F$ to greater than it:

$$\mathcal{E} < \mathcal{E}_F \;\; \to \;\; e^{-\text{big}} \;\; \to \;\; F(\mathcal{E}) \;\; \to \;\; 1$$

$$\mathcal{E} > \mathcal{E}_F \;\; \to \;\; e^{+\text{big}} \;\; \to \;\; F(\mathcal{E}) \;\; \to \;\; 0$$

The smaller kT is compared to $\mathcal{E}_F$, the more abrupt the transition of $(\mathcal{E}-\mathcal{E}_F)/kT$ from a big negative to a big positive number will be. So our result for complete degeneracy is quite simple:

If $\mathcal{E} < \mathcal{E}_F$ then $F(\mathcal{E}) = 1$, while for $\mathcal{E} > \mathcal{E}_F$ we set $F(\mathcal{E}) = 0$.

As a result, our expression for the pressure (equation 9) becomes

$$P \;=\; \frac{8\pi}{3h^3}\int_0^{p_F} \; v(p)\; p^3\; dp \;=\; \frac{8\pi}{3h^3}\int_0^{p_F} \frac{c^2 p^4\; dp}{\sqrt{p^2c^2 \;+\; m^2c^4}} \quad , \tag{11}$$

where we have used equation(3) for $v(p)$. But what is the value of the Fermi momentum, p_F ?. We go back to equation(8) for n, the number density of electrons, and use $F(\mathcal{E})$:

$$n \;=\; \frac{8\pi}{h^3}\int_0^{\infty} F(\mathcal{E})\; p^2\; dp \;=\; \frac{8\pi}{h^3}\int_0^{p_F} p^2\; dp \;=\; \frac{8\pi}{3h^3}\; p_F^3 \tag{12}$$

Solving for p_F, we find that

$$p_F \;=\; \left(\frac{3}{8\pi}\right)^{1/3} h\; n^{1/3}\; . \tag{13}$$

At this point, it is useful to introduce a new variable, $x = p/mc$, and, corresponding to the Fermi momentum, $x_F = p_F/mc$. The variable x is dimensionless, and when $x_F = 1$ we see that $p_F = mc$. This is the approximate value of momentum that marks the transition from non-relativistic to relativistic behavior. In terms of x, equation(13) becomes

$$x_F \;=\; \left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{h}{mc}\right)\; n^{1/3} \quad , \tag{14}$$

and equation(11) can be written

$$P \;=\; \frac{8\pi}{3h^3}\int_0^{x_F} \frac{c^2\; m^4c^4x^4\; mc\; dx}{\sqrt{m^2c^2x^2\; c^2 \;+\; m^2c^4}} \;=\; \frac{8\pi c^5 m^4}{3h^3}\;\int_0^{x_F} \frac{x^4\; dx}{\sqrt{1 \;+\; x^2}} \;=\; A\; f(x_F) \tag{15}$$

where

$$A \;=\; \frac{\pi}{3}\left(\frac{mc}{h}\right)^3\; mc^2 \;=\; 6.002\times 10^{22} \quad \text{dyne cm}^{-2} \tag{16}$$

and

$$f(x_F) \;=\; x_F(2x_F^2 \;-\; 3)(1 \;+\; x_F^2)^{1/2} \;+\; 3\; \sinh^{-1}(x_F) \tag{17}$$

This is a wonderful result, but unfortunately it's complicated and not very illuminating – how does this function behave? (By the way, it is customary at this point to just write x for x_F, even though x was introduced as a running variable.)

To get a better idea of what's going on, we can look at two limiting cases. First, lets consider the case where the degeneracy is <u>non-relativistic</u>. (This corresponds to $x_F << 1$.) Then $v(p) = p/m$. Inserting this into equation(9), we immediately obtain

$$P_{nr} \;=\; \frac{8\pi}{3mh^3}\int_0^{p_F} \; p^4\; dp \;=\; \frac{8\pi}{15mh^3}\; p_F^5 \tag{18}$$

Using equation(13), this becomes

$$P_{nr} \;=\; \frac{8\pi}{15mh^3}\;\left(\frac{3}{8\pi}\right)^{5/3}\; h^5\; n^{5/3} \;=\; \left(\frac{3}{\pi}\right)^{2/3}\;\frac{h^2}{20\; m}\; n^{5/3} \tag{19}$$

Since the number density of electrons will be proportional to the density ρ, this implies that $P_{nr} \propto \rho^{5/3}$. Compared to an ideal gas (where $P \propto \rho$), the degenerate gas is more resistant to compression (i.e., has a "stiffer" equation of state).

At the other extreme, consider the pressure of a highly relativistic degenerate electron gas. (This corresponds to $x_F >> 1$.) Then $v = c$, and equation(9) becomes

$$P_{rel} \;=\; \frac{8\pi c}{3h^3} \int_0^{p_F} p^3 \; dp \;=\; \frac{2\pi c}{3h^3}\, p_F^4 \tag{20}$$

$$P_{rel} \;=\; \frac{2\pi c}{3h^3} \left(\frac{3}{8\pi}\right)^{4/3} h^4 \; n^{4/3} \;=\; \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} n^{4/3} \quad . \tag{21}$$

Again, since n is proportional to ρ, we see that in the highly relativistic case, $P_{rel} \propto \rho^{4/3}$. Compared to the non-relativistic limit, this pressure is "softer". This is because the increase in the momentum of relativistic particles does not increase their velocity significantly (it's already nearly c). That means that the volume of the cylinder in the diagram on page 2 doesn't increase with p. It turns out that the difference between $\rho^{5/3}$ and $\rho^{4/3}$ makes all the difference in the world – it leads to a limit to the mass of a white dwarf and thus plays a key role in stellar evolution.

Finally, we note that equation(17) is a function that varies smoothly between the non-relativistic and highly relativistic limits we have obtained above.

Now for use in the equations of stellar structure, we want to express the pressure as a function of the density ρ rather than the number of electrons n. We can assume that the gas is fully ionized. We introduce the quantity μ_e , the mean molecular weight per electron. That is, how many units of atomic mass are there per electron? For example, for hydrogen, there is one proton for each electron, thus one unit of atomic weight per electron and hence $\mu_e = 1$. But if we had ionized helium, there would be one nucleus of weight 4 for each 2 electrons, and hence $\mu_e = 2$. For carbon, the weight of the nucleus is 12 and it contributes 6 electrons, so again $\mu_e = 2$. We can calculate μ_e for any mixture of elements. But, in fact, white dwarfs never contain hydrogen in their interiors; they are composed of He, C, O, or perhaps some Ne. And for all these, we have $\mu_e = 2$.

Now number of electrons cm^{-3} will be the density in g cm^{-3} divided by $\mu_e \; m_H$, the number of grams per electron:

$$n_e \;=\; \frac{\rho}{\mu_e m_H} \;=\; \frac{N_A}{\mu_e}\, \rho \quad , \tag{22}$$

where m_H is the mass of the hydrogen atom and N_A is the Avogadro number. So we finally write equations (19) and (21) as

$$P_{nr} \;=\; \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20\; m} \left(\frac{N_A}{\mu_e}\right)^{5/3} \rho^{5/3} \tag{23}$$

and

$$P_{rel} \;=\; \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{N_A}{\mu_e}\right)^{4/3} \rho^{4/3} \quad . \tag{24}$$