1. The Equation of Hydrostatic Equilibrium and Scale Heights

Consider the cylindrical volume of gas shown in Fig. 1. The mass of the cylinder is $Volume \times \rho \;=\; dA \times dr \times \rho$. Thus the gravitational force on the volume is $F_g \;=\; -g\,\rho\,dr\,dA$. If the cylinder is to remain at rest, F_g must be counterbalanced by the difference in the pressure forces on the two faces: $(dA\;P(r+dr)) - (dA\;P(r)) \;=\; dA\;dP$. Equating these forces gives us the **equation of hydrostatic equilibrium**:

$$\frac{dP}{dr} \;=\; -\,g\,\rho \tag{1}$$

Now in general, P will be a function of both ρ and T, and thus we will need an equation for dT/dr to make further progress. But there are a number of important cases in astrophysics where the pressure P is a function of ρ alone. For instance, it is sometimes the case that $P \;=\; \rho^\gamma$, so that the hydrostatic equilibrium equation becomes a function of ρ alone. We will consider perhaps the simplest case, where the temperature is constant. If the fluid is an ideal gas, then the pressure, density and temperature are related by the equation of state:

$$P \;=\; nkT \;=\; \frac{\rho}{m}\,kT \;=\; \frac{N_A k}{\mu}\,\rho\,T \tag{2}$$

where N_A is Avagadro's number and μ is the mean molecular weight in atomic units. So we can write the equation of hydrostatic equilibrium as

$$\frac{1}{\rho}\frac{d}{dr}\left(\frac{kT}{m}\rho\right) \;=\; -\,g \tag{3}$$

and if T is constant, we can take it outside the differential

$$\left(\frac{kT}{m}\right)\frac{1}{\rho}\frac{d\rho}{dr} \;=\; -\,g \tag{4}$$

and moving the constant terms to the r.h.s. we have

$$\frac{1}{\rho}\frac{d\rho}{dr} \;=\; \frac{d\ln\rho}{dr} \;=\; -\,\frac{gm}{kT} \tag{5}$$

We can integrate this to obtain

$$\ln\rho \;=\; -\,\frac{gm}{kT}\,r \;+\; const \tag{6}$$

Since $e^{\ln\rho} \;=\; \rho$, we have

$$\rho \;=\; C\;\exp\left(-\frac{gm}{kT}\,r\right) \tag{7}$$

where C is the (exponentiated) constant of integration. To set this constant, we suppose we know the value $\rho(r_0)$ at some level r_0. Then we see that we must have $C = \rho(r_0)e^{gmr_0/kT}$, and our solution becomes

$$\rho \;=\; \rho_0\;\exp\left[-\frac{gm}{kT}\,(r-r_0)\right] \tag{8}$$

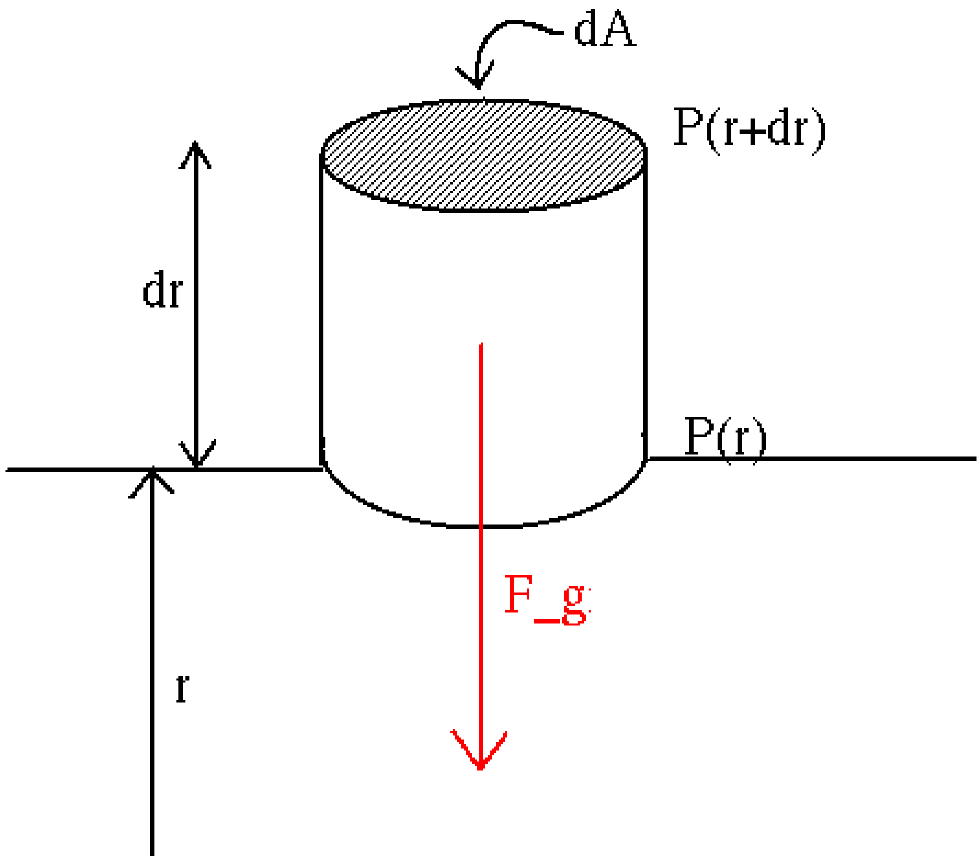


Fig. 1.— The forces on a volume element in hydrostatic equilibrium.

In view of equation (8) we are led to define the *scale height*, H, as

$$H = \frac{kT}{gm} = \frac{c_i^2}{g} \quad , \tag{9}$$

where c_i is the *isothermal sound speed* given by

$$c_i^2 = \frac{kT}{m} = \frac{N_A k}{\mu} T \, . \tag{10}$$

(Note that the "normal" speed of sound is the *adiabatic sound speed*, c_a , where $c_a^2 = \gamma c_i^2$. This is because when a sound wave compresses a gas, it will usually heat it adiabatically, so that T is not constant.) With this definition of H, the density in an isothermal atmosphere under constant gravity is just

$$\rho(r) = \rho_0 \exp\left[-\frac{(r - r_0)}{H}\right] \tag{11}$$

As a familiar example, consider the Earth's atmosphere. We have $g = 980$ cm s^{-2}, $T \sim 280$ K, a composition mostly N_2, so that $m = 28 \; m_H = 4.7 \times 10^{-23}$ g, and finally $k = 1.38 \times 10^{-16}$. Thus the scale height of the Earth's atmosphere (to the extent that the temperature is constant) is

$$H = \frac{(1.38 \times 10^{-16})(280)}{(980)(4.7 \times 10^{-23})} = 839,000 \;\; \text{cm} \quad = 8.4 \;\; \text{km}, \tag{12}$$

The peak of Mount Everest is about 8.85 km (29,000 ft) above sea level. From equation (11) we see that the air density (and pressure) at the summit will be $\exp(-8.85/H) = \exp(-1.05) = 0.35$ times that at sea level. Humans can only survive at pressures below half an atmosphere for short periods of time.

2. Hydrostatic Equilibrium in Isothermal Thin Disks

Another application of hydrostatic equilibrium is the case of a disk of gas orbiting a massive central object. This could be a proto-planetary disk in orbit about a young proto-star, an accretion disk about a neutron star or a black hole, or even the interstellar gas in the disk of a spiral galaxy. The important point for this application is that the self-gravity of the orbiting gas is negligible compared to the gradient in the gravitational field of the central object. Let the coordinate system be cylindrical, with distance R from the disk axis and with z the height above the plane. Then the distance from the central mass M_* to the parcel of gas at (R, z) is just $r = (R^2 + z^2)^{1/2}$ and the gravitational potential is

$$\Phi(R, z) = -\frac{GM_*}{r} = -\frac{GM_*}{(R^2 + z^2)^{1/2}} \tag{13}$$

Now, unless $z = 0$, the gravitational force will not be entirely in the disk plane, but will be directed towards M_* and thus have a component in the z direction. But the parcel of gas cannot follow an inclined orbit about M_*; instead it must orbit parallel to the disk plane with the other parcels of gas having different z but with the same R. Thus the parcels of gas in this co-moving ring of gas will feel a gravitational force in the z-direction given by

$$g_z \;=\; -\frac{\partial \Phi}{\partial z} \;=\; -\frac{GM_*}{(R^2 \;+\; z^2)^{3/2}}\, z \tag{14}$$

For a thin disk, $z << R$, and therefore the effective gravity is

$$g_z \;=\; -\frac{GM_*}{R^3}\, z \;=\; -\,\Omega^2\, z \tag{15}$$

Note that this force is a tidal (differential) force and thus g_z falls off as R^{-3}. Ω represents the angular velocity of particles in Keplerian orbits at distance R. This last form arises because $\Omega = v_{cir}/R$ and the circular orbital velocity v_{cir} is

$$v_{cir}^2 \;=\; \frac{GM_*}{R} \qquad \text{so that} \qquad \Omega^2 \;=\; \frac{v_{cir}^2}{R^2} \;=\; \frac{GM_*}{R^3} \tag{16}$$

Now, returning to the equation of hydrostatic equilibrium, we insert g_z to obtain

$$\frac{1}{\rho}\frac{dP}{dz} \;=\; -\,g_z \;=\; -\frac{GM_*}{R^3}\, z \tag{17}$$

We will assume temperature to be constant in the z-direction (it can – and likely will – vary with R). Then, from equations (2) and (10), we can write the pressure as $P \;=\; c_i^2\,\rho$, and we have

$$c_i^2\,\frac{1}{\rho}\frac{d\rho}{dz} \;=\; -\,\Omega^2\, z \tag{18}$$

We can immediately integrate this:

$$\ln \rho \;=\; -\left(\frac{\Omega}{c_i}\right)^2 \frac{z^2}{2} \;+\; const \tag{19}$$

and exponentiating we obtain

$$\rho(z) \;=\; \rho_0\; \exp\left[-\left(\frac{\Omega}{c_i}\right)^2 \frac{z^2}{2}\right] \;=\; \rho_0\; \exp\left[-\frac{z^2}{2H^2}\right] \tag{20}$$

where we let ρ_0 be the density at the mid-plane where $z = 0$. We see that in this case, the scale height H is given by

$$H \;=\; \frac{c_i}{\Omega} \;=\; \left(\frac{kT}{m}\right)^{1/2} \left(\frac{GM_*}{R^3}\right)^{-1/2} . \tag{21}$$

We see that $H \propto R^{3/2}$, so we might expect the disk to increase in thickness as we go outwards, but to make a firm prediction, we must have some idea of how T varies with R. For example, in the classical theory of accretion disks, it turns out that $T \propto R^{-3/4}$. So in this case $H \propto R^{-3/8} \cdot R^{3/2} = R^{9/8}$, and indeed the disk is expected to flare up as we move outwards.

3. The Theory of Isothermal Spheres

So far we have looked at cases where the gravitational force was external and given, but now we want to consider the simplest case where the gravitational force is due to the gas itself. We assume a spherical distribution, so that the gravitational force at any point r will be due to the mass M_r contained within the sphere of radius r. Thus the equation of hydrostatic equilibrium will be

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\,\rho \tag{22}$$

Since the density is not constant, we must treat M_r with care. The best we can do is write down a differential equation for M_r: If we are at some radius r and we move outwards to $r+dr$, we have added a shell of volume $dV = 4\pi r^2 dr$. The added mass is thus $dM_r = \rho(r)dV$. So the equation for M_r is

$$\frac{dM_r}{dr} = 4\pi\, r^2\, \rho \ . \tag{23}$$

Let us write equation (22) as

$$\frac{r^2}{\rho}\,\frac{dP}{dr} = -\,GM_r \tag{24}$$

and take its derivative, making use of eqn (23):

$$\frac{d}{dr}\left[\frac{r^2}{\rho}\,\frac{dP}{dr}\right] = -\,G\,\frac{dM_r}{dr} = -\,4\pi G\, r^2\, \rho \ . \tag{25}$$

Taking r^2 to the other side, we have a second order differential equation:

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\,\frac{dP}{dr}\right] = -\,4\pi\, G\, \rho \ . \tag{26}$$

In general, $P = P(\rho, T)$ and the equation above must be coupled with an equation for dT/dr from energy flow arguments. But there are some important cases in astrophysics where $P = P(\rho)$ only. We can then solve for the structure $P(r)$ and $\rho(r)$. An important example of this are the so-called polytropes, which are the solutions which follow when $P \propto \rho^\gamma$, where γ is a constant – in some cases the ratio of specific heats.

We will consider what may be the simplest case, that of an *isothermal ideal gas*. Then once again $P = c_i^2\, \rho$ where c_i is the constant isothermal sound speed and equation (26) becomes

$$\frac{c_i^2}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\,\frac{d\rho}{dr}\right] = -\,4\pi\, G\, \rho \ . \tag{27}$$

Since $d\ln\rho/dr = (1/\rho)(d\rho/dr)$, we write this as

$$\frac{c_i^2}{4\pi G}\;\frac{1}{r^2}\frac{d}{dr}\left[r^2\,\frac{d\ln\rho}{dr}\right] = -\,\rho \ . \tag{28}$$

We now introduce the variable w through $\rho = \rho_0\, e^{-w}$. Then $\ln\rho = \ln\rho_0 - w$ so that $d\ln\rho/dr = -dw/dr$.

Equation (28) then becomes

$$\left(\frac{c_i^2}{4\pi G\rho_0}\right) \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dw}{dr}\right] = e^{-w} \quad . \tag{29}$$

Finally, let us introduce a new radial coordinate z scaled by some constant A: $r = Az$, $dr = A\, dz$. Then we see that

$$\left(\frac{c_i^2}{4\pi G\rho_0}\right) \frac{1}{A^2} \frac{1}{z^2}\frac{d}{dz}\left[z^2 \frac{dw}{dz}\right] = e^{-w} \quad . \tag{30}$$

We thus are led to set $A^2 = c_i^2/(4\pi G\rho_0)$. With this choice our equation becomes

$$\boxed{\frac{1}{z^2}\frac{d}{dz}\left[z^2 \frac{dw}{dz}\right] = e^{-w}} \tag{31}$$

where

$$r = \frac{c_i}{(4\pi G\rho_0)^{1/2}}\, z \quad \text{and} \quad \rho = \rho_0\, e^{-w} \quad . \tag{32}$$

If we take as a boundary condition $w = 0$ at the center $r = z = 0$, then we see that ρ_0 is the central density. Also, as we approach the center, M_r goes to zero as r^3, the pressure gradient must vanish, and it follows that a second boundary condition is $dw/dz = 0$ at $z = 0$.

With these boundary conditions, equation (31) has no analytic solution, but must be integrated numerically. We also see that we cannot start the numerical integration at $z = 0$ because of the $1/z^2$ term. By assuming that $w(z)$ has the form $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ we can put this into equation (31) and, by equating like powers of z, find a series expansion:

$$w(z) = \frac{1}{6}\, z^2 - \frac{1}{5 \cdot 4!}\, z^4 + \frac{8}{21 \cdot 6!}\, z^6 - \cdots \tag{33}$$

This series (and its derivative) can be used to evaluate $w(z)$ and dw/dz for some $z << 1$, say $z = 0.01$. (The series is useless for $z > 1$.)

The usual approach to the numerical integration of a second order differential equation is to rewrite it as a pair of first order equations. Thus we define the first derivative as a new variable y:

$$\frac{dw}{dz} = y \tag{34}$$

and inserting this into eqn(31) above we have

$$\frac{dy}{dz} = e^{-w} - \frac{2}{z}\, y \tag{35}$$

The solution thus obtained has $w(z)$ monotonically increasing as $z \to \infty$, and thus $\rho(z) \propto e^{-w}$ decreases monotonically – but $\rho(z)$ never reaches zero. This solution – up to $z \sim 7.5$ – is shown in Fig. 2.

Not only does $\rho(z)$ extend to infinity, but even worse, for very large z, $\rho(z) \propto z^{-2}$, so the mass inside a given radius, M_r, does not converge: the solution has infinite mass! So we

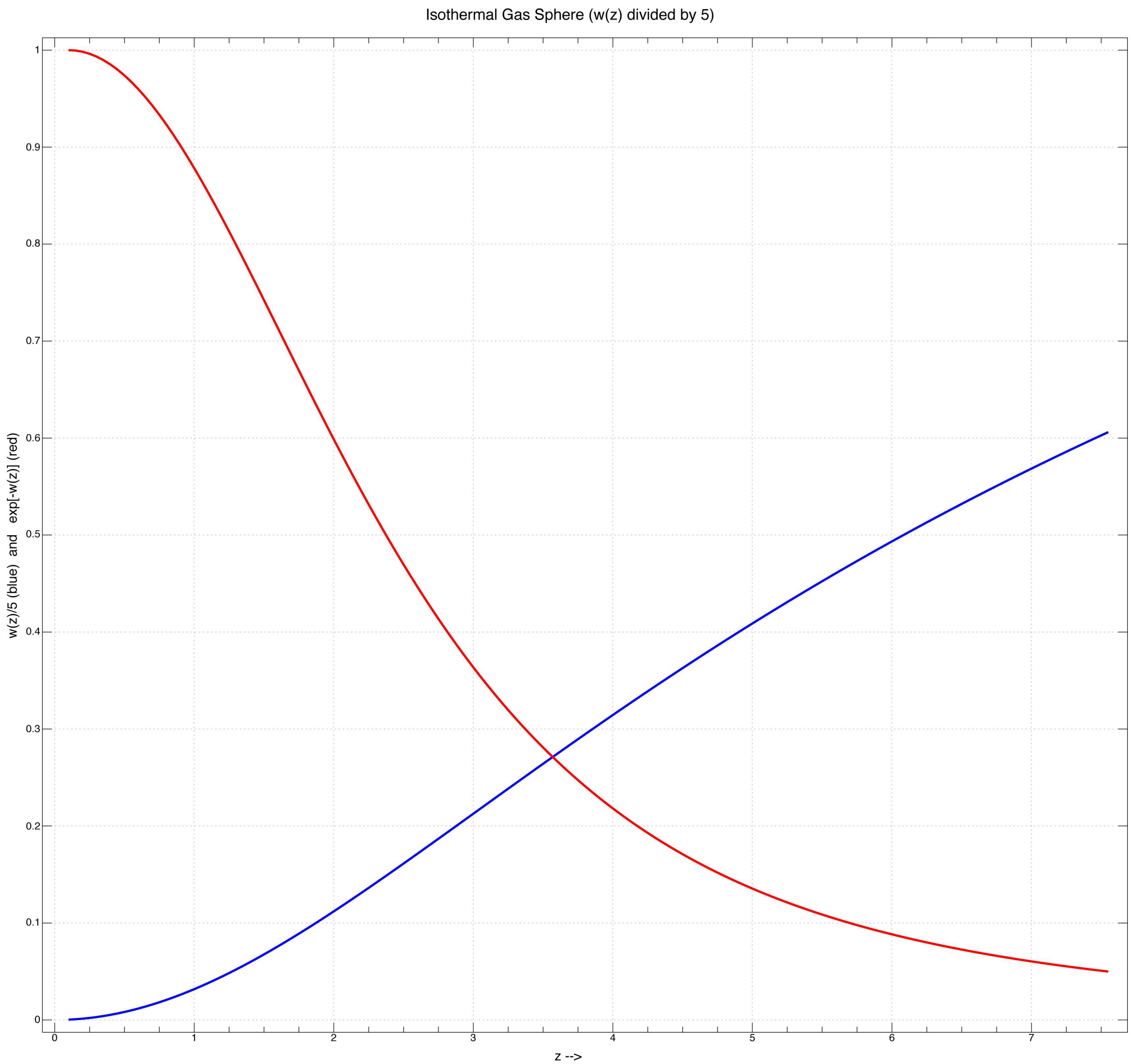


Fig. 2.— The solution for the isothermal sphere for $0 < z < 7.5$. The red curve is exp(-w) which is proportional to the density.

may well ask if such a solution has any relevance to actual physical situations. The answer is yes. But we must *truncate* the solution at some point z_{edge}, at which point the gas will have a non-zero pressure $P_{edge} = c_i^2 \rho(z_{edge})$, and this pressure must be matched by an external pressure. For example, consider a cold, dense cloud of H_2 surrounded by a hot, low density interstellar medium: this medium can exert the pressure P_{edge} and the cold gas inside may have the density distribution of the truncated isothermal sphere.

We said that there is no analytic solution of the isothermal equation with our boundary conditions $w(0) = 0$ and $w'(0) = 0$. There is, however, a solution if we let $\rho(0) \to \infty$. The solution is simply

$$w \;=\; 2\ln(z) \;-\; \ln(2) \qquad \text{so that} \qquad e^{-w} \;=\; \frac{2}{z^2} \tag{36}$$

which is easily verified by substituting this $w(z)$ into eqn(31). In terms of r, this is

$$\rho(r) \;=\; \frac{c_i^2}{2\pi G}\,\frac{1}{r^2} \tag{37}$$

This solution is called the "singular isothermal sphere" (SIS). It can be shown that for large z the regular (numeric) solution approaches the SIS solution more and more closely. We can see this in Fig. 3 where we plot both solutions on a log-log plot. But we get a hint that things are not simple when we notice that the regular solution does not approach the singular solution asymptotically, but rather oscillates around it, crossing first at $z \sim 1.7$ and then crossing back at $z \sim 15.7$, etc. The behavior of the SIS density near the center may seem absurd, but at least the SIS mass M_r is well behaved everywhere. Indeed, inserting eqn(37) into eqn(23) and integrating, we find that for the SIS solution, $M_r \;=\; (2c_i^2/G)\; r$ (or in terms of z , $M_z = 8\pi\; z$).

However, although the truncated regular solutions are all solutions, *they are not all stable.* If we truncate the solution for small z, the enclosed gas doesn't vary much in density, so if we compress it, it pushes back like gas in a balloon. But for spheres truncated at larger z, with a higher ratio of $\rho(center)/\rho(edge)$, the gravitational energy is more important, and at some point the disturbed sphere will collapse. This was discovered independently by Ebert(1955) and by Bonnor(1956). It is easy to show from the numerical solution if we simply plot the pressure vs. the volume of partial spheres while holding the mass constant. To do this we first look at the mass of the sphere truncated at some edge $r = R$. Using equation (32) we have

$$M(R) \;=\; \int_0^R 4\pi r^2 \rho(r) dr \;=\; 4\pi\rho_0 \int_0^R e^{-w} r^2 dr \;=\; \frac{4\pi\rho_0\; c_i^3}{(4\pi G\rho_0)^{3/2}} \int_0^{z(R)} e^{-w} z^2 dz \tag{38}$$

But from equation (31) we have that

$$\frac{d}{dz}\left[z^2\,\frac{dw}{dz}\right] \;=\; e^{-w}\; z^2 \tag{39}$$

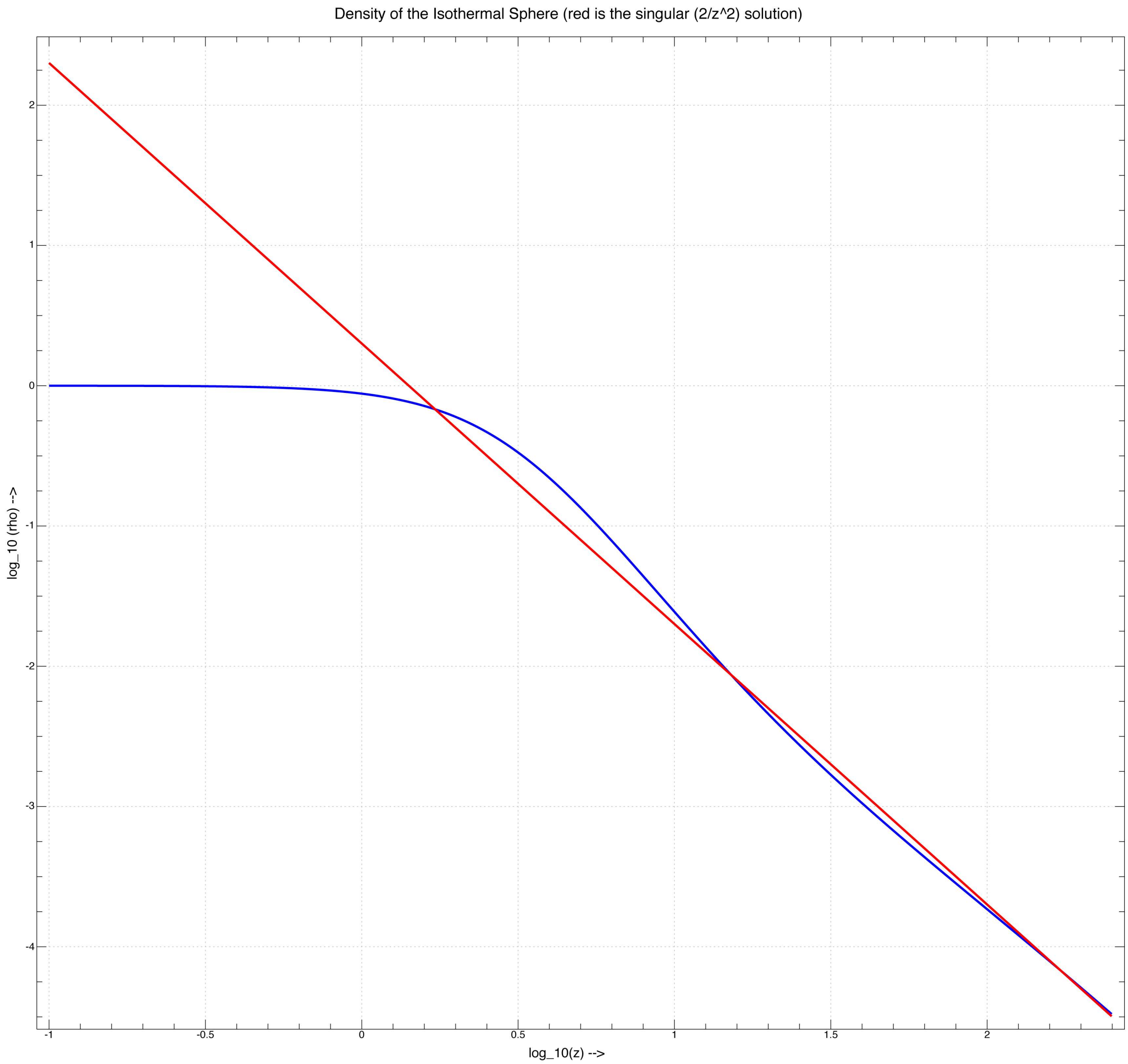


Fig. 3.— The log density of the isothermal sphere for $0.1 < z < 250$ (blue curve). The red curve is the singular (SIS) solution $2\ z^{-2}$.

so we can evaluate the integral:

$$\int_0^z e^{-w} z^2 dz \;=\; \int_0^z d\left(z^2\,\frac{dw}{dz}\right) \;=\; \left[z^2\,\frac{dw}{dz}\right]_{z(R)} \tag{40}$$

The mass is therefore

$$M(R) \;=\; \frac{c_i^3}{(4\pi G^3 \rho_0)^{1/2}}\;\left[z^2\,\frac{dw}{dz}\right]_{z(R)} \tag{41}$$

Solving for the central density ρ_0 we find

$$\rho_0 \;=\; \frac{c_i^6}{4\pi G^3}\frac{1}{M^2}\;z^4\left(\frac{dw}{dz}\right) \tag{42}$$

which in turns lets us write the pressure at the edge $r = R$:

$$P_R(z) \;=\; c_i^2\;\rho \;=\; c_i^2\;\rho_0\;e^{-w} \;=\; \frac{1}{4\pi G^3}\;\frac{c_i^8}{M^2}\;z^4\;\left(\frac{dw}{dz}\right)^2\;e^{-w} \tag{43}$$

Again, from the scaling factor (equation 32), we can write

$$\rho_0 \;=\; \frac{c_i^2}{4\pi G}\;\frac{z^2}{r^2} \tag{44}$$

Equating this with ρ_0 from equation (42) and solving for $r = R$, we find

$$R \;=\; \frac{GM}{c_i^2}\;\left(z\;\frac{dw}{dz}\right)^{-1} \quad \text{so that the volume is} \quad V_R(z) \;=\; \frac{4\pi}{3}\;\left(\frac{GM}{c_i^2}\right)^3\;\left(z\;\frac{dw}{dz}\right)^{-3} \tag{45}$$

Thus, as we vary z and use the values $w(z)$ and dw/dz from our numerical solution of equation (31), equations (43) and (45) give us the volume V_R and boundary pressure P_R of a series of truncated isothermal spheres – called Bonnor-Ebert spheres – of constant mass M. Ignoring the constant factors, in Fig. 4 we plot P_R as a function of V_R. The spheres become unstable at the point at which P reaches its maximum. Beyond this point, $\partial P/\partial V > 0$, and compression to a smaller volume results in a *decrease* in the boundary pressure and hence to collapse. This maximum of P occurs at

$$z_{crit} = 6.45\;, \quad \text{where} \quad z^4\left(\frac{dw}{dz}\right)^2 e^{-w} \;=\; 17.5635 \quad \text{and} \quad \left(z\;\frac{dw}{dz}\right)^{-1} \;=\; 0.4108\;. \tag{46}$$

In addition, the critical value of $\rho_0/\rho_{edge}(=\;\; e^{-w})$ is 14.04. Fig. 5 shows a sequence of density profiles, including the critical sphere. (We note that even though the slope becomes negative again beyond the first maximum of P, there will be internal ($r < R$) unstable spheres, so there really are no stable solutions beyond this point.) Putting the values of (46) into equations (43) and (45), we obtain

$$P_{R,max} \;=\; 1.398\;\frac{c_i^8}{G^3 M^2} \quad \text{and} \quad R_{min} \;=\; 0.4108\;\frac{GM}{c_i^2}\;. \tag{47}$$

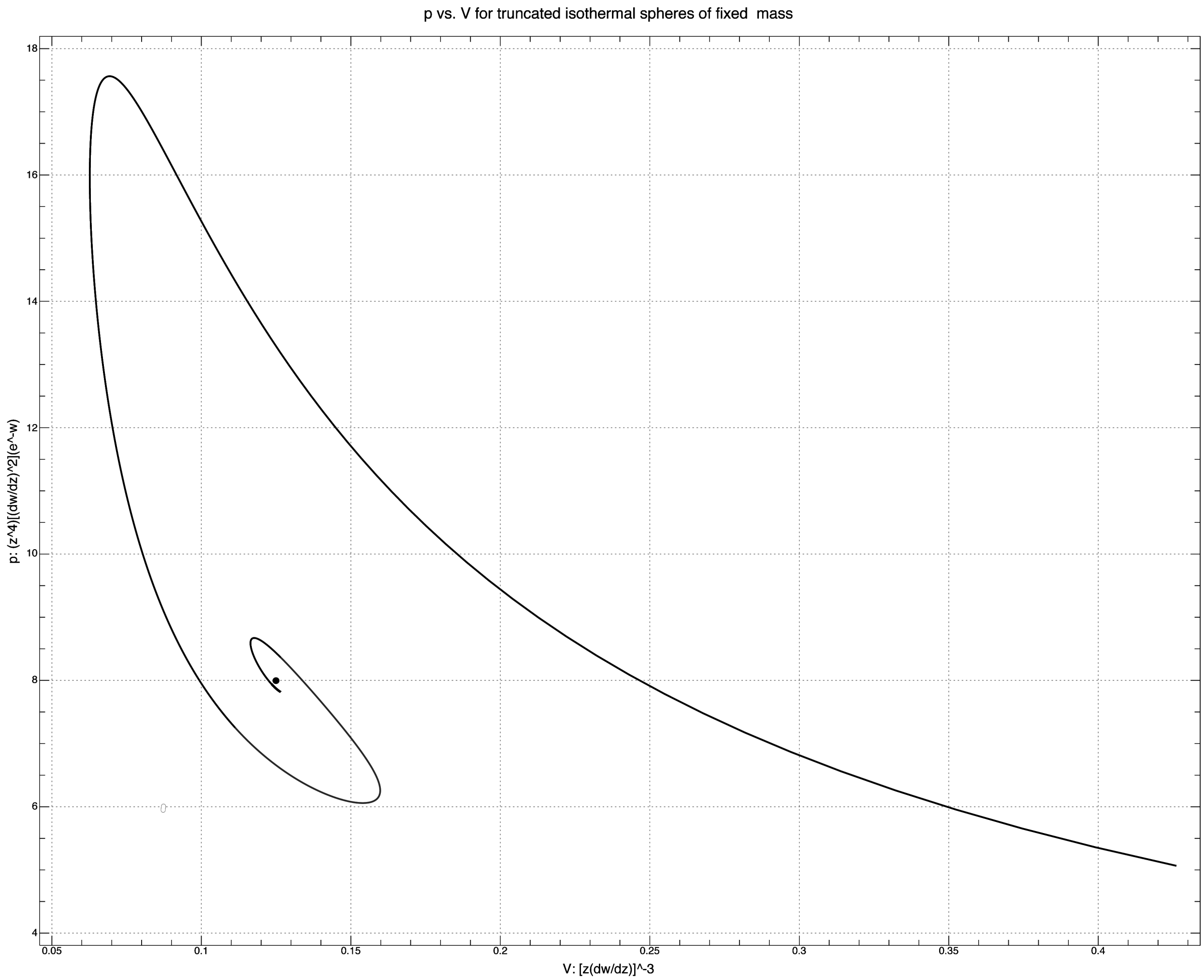
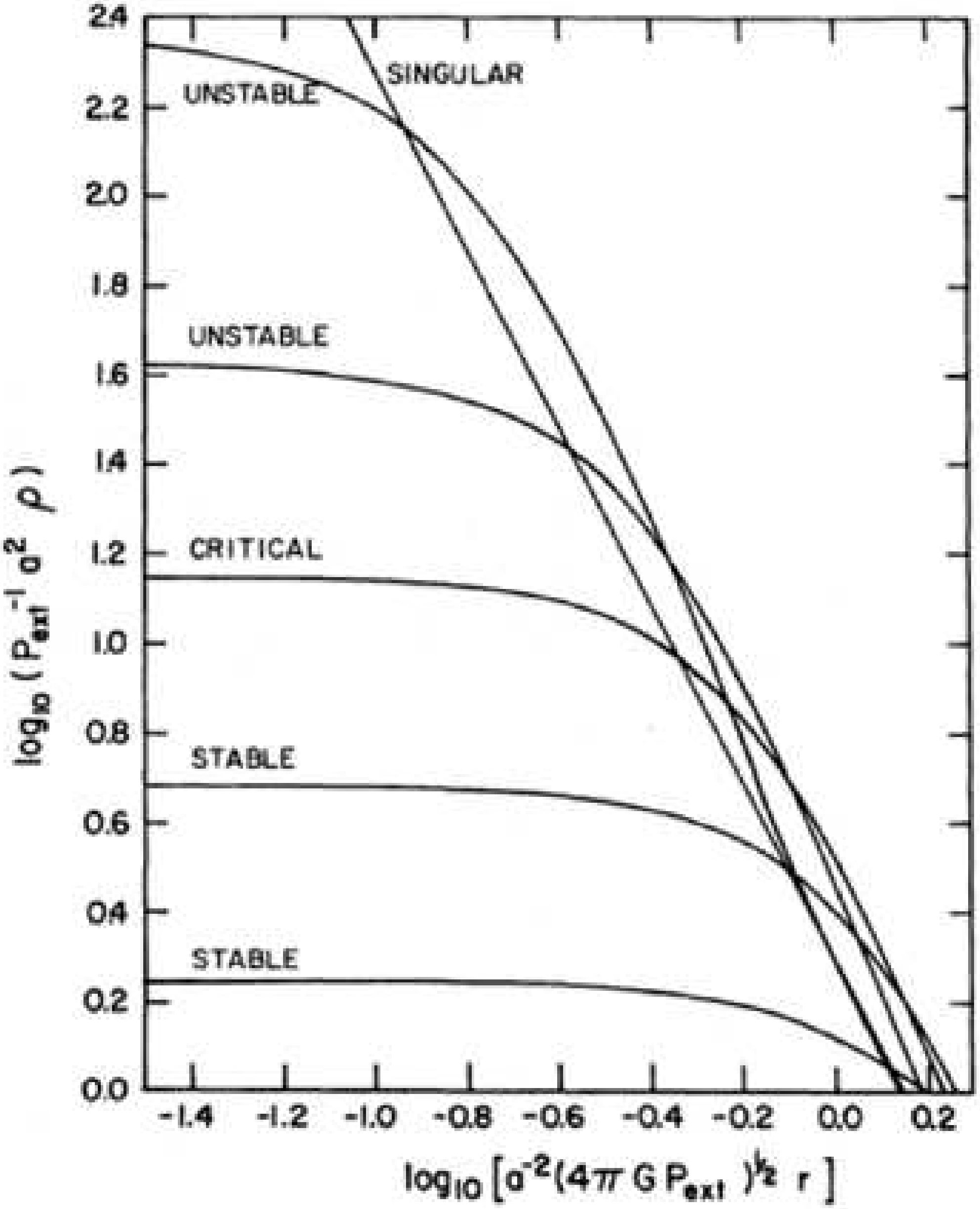


Fig. 4.— P vs V for isothermal spheres truncated at different radii but having the same mass. As $z \to \infty$, the curve spirals around the dot at (V=$\frac{1}{8}$,P=8), which marks the SIS solution.



—Density distributions of bounded isothermal spheres. The outer radius of each sphere is given by the intercept of the corresponding curve with the abscissa. The curve marked "critical" denotes the sphere with the maximum mass consistent with hydrostatic equilibrium at a given external pressure. Hydrostatic spheres which are less centrally concentrated than the critical Bonnor-Ebert state are gravitationally stable; those which are more centrally concentrated are gravitationally unstable. In the limit of infinite central concentration, the latter spheres approach the singular solution.

Fig. 5.— Stable and Unstable Isothermal Spheres. The limit of stability is reached when $\rho_0/\rho_{edge} = 14.04$

We may rearrange the first expression to give the maximum mass of a cloud with a given sound speed (temperature), subjected to an external pressure P_R or boundary density ρ_{edge}:

$$M_{crit} = 1.182 \frac{c_i^4}{(G^3 P_R)^{1/2}} = 1.182 \frac{c_i^3}{(G^3 \rho_{edge})^{1/2}} . \tag{48}$$

Let's compare this critical sphere with the well known *Jeans mass*:

$$M_J = \frac{\pi^{5/2}}{6} \frac{c_i^3}{(G^3 \rho)^{1/2}} = 2.916 \frac{c_i^3}{(G^3 \rho)^{1/2}} , \tag{49}$$

where in this case the medium is assumed to have a uniform density ρ. To compare the two results, we should express equation (48) in terms of the mean density $\bar{\rho} = M/V$ of the Bonnor-Ebert sphere. We easily find from (47) and (48) that $\rho_{edge} = 0.4057\ \bar{\rho}$ (and also $\rho_0 = 5.696\ \bar{\rho}$). As a result equation (48) for the critical Bonnor-Ebert sphere can also be written as

$$M_{crit} = 1.86 \frac{c_i^3}{(G^3 \bar{\rho})^{1/2}} . \tag{50}$$

We see that Jeans mass is about 60% larger – this is not surprising as the gravitational potential energy of the isothermal sphere is higher than the potential of the same mass and mean density when distributed uniformly.

As an example, consider a cold H_2 cloud surrounded by a hotter, lower density medium. Take $T = 20$ K, and $\mu = 2.35$ for molecular hydrogen plus helium. Then $c_i^2 = (kT/\mu m_H) = 7 \times 10^8$ $(\mathrm{cm/s})^2$ ($c_i = 0.265$ km/s). The pressure at the edge is $P = nkT$, and let us take $n_{edge} = 10^5$ cm^{-3} ($n_{core} = 1.4 \times 10^6$) Then the pressure is $P_R = 2.76 \times 10^{-10}$ dyn cm^{-2}. Putting these values into equation (48), we find $M_{crit} = 2 \times 10^{33}$ g $= 1\ M_\odot$. The corresponding cloud radius is $R_{min} = 8 \times 10^{16}$ cm $= 0.026$ pc $= 5300$ AU.

Other systems can be modeled as isothermal spheres. Globular clusters are ~ 10 billion years old and have had time for stellar encounters to set up a spatially uniform velocity distribution. It turns out that the density of stars in these objects is often well described by the isothermal profile.

Finally, we note that the velocity of an object in a circular orbit within an SIS mass distribution will be constant: $v_{cir} = (GM_r/r)^{1/2} = \sqrt{2}\ c_i$. Recall that the rotation curve of our Galaxy (and other spiral galaxies) is flat beyond the sun, and that this behavior is generally attributed to a halo of dark matter. This dark matter then has to have the density profile $\rho \propto r^{-2}$. The likely explanation is that the dark matter halo has the profile of an isothermal sphere. Since most of the mass of the universe (aside from the dark energy) is dark matter, equation (31), the isothermal sphere, may describe most of the cosmos!

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