ASTR 320 - Theoretical Astrophysics

Spring 2013

Homework Assignment No. 1

1. It is easy to show that for an ideal gas, where the internal energy $u(T)$ is a function of T only, we can write

$$dQ \;=\; c_V\,dT \;+\; (c_P - c_V)\,\frac{T}{V}\,dV$$

where c_V and c_P are the specific heats at constant volume and pressure, respectively.

   (a) What does this equation become for an *adiabatic* change? In this case, show how a logarithmic change in temperature dT/T is related to the change in specific volume dV/V. Write your expression in terms of the ratio of specific heats $\gamma = c_P/c_V$.

   (b) For a monatomic ideal gas, we saw that $\gamma = 5/3$. Suppose we have a volume V of monatomic gas (helium, for example) in an insulated cylinder, and that we compress it to one half its previous volume, $\frac{1}{2}V$. Suppose further that, before compression, the temperature of the helium is $T_0 = 27^oC = 300^oK$. (Which temperature scale do you want to use for this calculation?) Find the temperature $T_1^{(m)}$ after compression.

   (c) Suppose that instead of helium, the gas is nitrogen which (at these temperatures) is a diatomic gas. If the ratio of specific heats for this gas is $\gamma = 7/5$, find the temperature after compression, $T_1^{(d)}$, under the same conditions as in (b) above.

   (d) Is $T_1^{(m)}$ greater or less than $T_1^{(d)}$? Give a physical explanation for this result.

2. Under conditions of thermal equilibrium, the distribution of photons with momentum obeys the equation

$$dn(\vec{p}) \;=\; g(\vec{p})\;d^3p \qquad \text{where} \qquad g(\vec{p}) \;=\; \frac{2/h^3}{e^{\frac{c|\vec{p}|}{kT}} \;-\; 1}$$

   (a) Integrate over momenta to evaluate $n = \int g(\vec{p}) d^3p$ in terms of fundamental constants and the temperature. (Do this in spherical coordinates. You will have to look up the integral which results.)

   Compute the mean number of photons per unit volume due to the Cosmic Background Radiation (CBR) now ($T = 2.73$K) and at the epoch of radiation-matter decoupling when the temperature was a factor of ~ 1400 higher.

   (b) The average total photon energy per unit volume is given by

$$u \;=\; \int c\,|\vec{p}|\;g(\vec{p})\;d^3p$$

   We showed in class that u must be given by $u = a_{rad}T^4$ for some constant factor a_{rad}. Evaluate the integral for u to find an expression for a_{rad} in terms of fundamental constants. (Check with the tabulated value.)

   Compute u for the CBR now and at the time of decoupling.

3. The Friedmann equation is given by (e.g., Maoz, p 195; Choudhuri, p 309, where R→a):

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

For the matter-dominated case $\rho = \rho_0(R_0/R)^3,$ so we can write it as

$$\left(\dot{R}\right)^2 - 2\left(\frac{A}{R}\right) = -kc^2 \quad \text{where} \quad A = \frac{4}{3}\pi R_0^3\, G\rho_0$$

The solution for a closed universe ($k = 1$) can be written in terms of a parameter x as

$$R(x) = \frac{A}{kc^2}\,(1 - \cos(x))$$

and for the time t,

$$t(x) = \frac{A}{(kc^2)^{3/2}}\,(x - \sin(x))$$

(a) Use $\dot{R} = dR/dt = (dR/dx)/(dt/dx)$ to show that these equations do in fact satisfy the second form of the Friedmann equation given above.

(b) Use the parametric solution above to plot $R(x)$ vs $t(x)$ over the range of $0 < x < 2\pi$. Calculate many points so you can plot a smooth curve. (It is best to use a software package to make the plot.) At what time t does R reach its maximum value?

Due: 7 February 2013