Problem Set No. 4 - solutions.

1. The Lane-Emden equation is \( \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Theta}{d\xi} \right) = -\Theta^n \)

(a) To show that \( \Theta(\xi) = \sin \xi / \xi \) is a solution, we evaluate

\[
\frac{d\Theta}{d\xi} = \frac{\cos \xi}{\xi} - \frac{\sin \xi}{\xi^2}, \quad \text{so} \quad \left( \xi^2 \frac{d\Theta}{d\xi} \right) = \xi \cos \xi - \sin \xi
\]

Next, \( \frac{d}{d\xi} \left( \xi^2 \frac{d\Theta}{d\xi} \right) = \cos \xi - \xi \sin \xi - \cos \xi = -\xi \sin \xi \)

So the l.h.s. is just \( \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Theta}{d\xi} \right) = -\frac{\sin \xi}{\xi} \)

But the r.h.s. is \(-\Theta^n = -\Theta^1 = -\frac{\sin \xi}{\xi} \), so the L-E equation is satisfied.

Use the expansion \( \sin \xi = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \cdots \)

to obtain

\[
\Theta(\xi) = \frac{\sin \xi}{\xi} = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} - \frac{\xi^6}{5040} + \cdots
\]

Clearly, \( \Theta(\xi = 0) = 1 \) and \( \frac{d\Theta}{d\xi} = -\frac{1}{3} \xi + \frac{1}{30} \xi^3 - \frac{1}{840} \xi^5 + \cdots \)

So \( \frac{d\Theta}{d\xi} |_{\xi = 0} = 0 \) at \( \xi = 0 \).

(b) We gave the expansion above. For \( n=1 \) we have

\[
\Theta(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{1}{120} \xi^4 - \frac{1}{15120} \xi^6 + \cdots \quad \text{but} \quad \frac{3}{15120} = \frac{1}{5040}
\]

so the series are identical.

(c) \( \frac{d\Theta}{d\xi} |_{\xi = \pi} = \left[ \frac{\cos \xi}{\xi} - \frac{\sin \xi}{\xi^2} \right] = \left[ \frac{-1}{\pi} - \frac{0}{\pi^2} \right] = -\frac{1}{\pi} \)

(d) \( D_n = -\left[ \frac{3}{\pi} \left( -\frac{1}{\pi} \right) \right]^{-1} = \left( \frac{3}{\pi^2} \right)^{-1} = \frac{\pi^2}{3} = 3.18387 \)
(2) (a) Combining the unnumbered equation between (1.5) and (1.6) with equation (1.6) we have

\[ E_{GR} = \int_0^R 4\pi r^2 \rho \frac{dP}{dr} \, d\tau \]

We are considering the model where \( \frac{dP}{dr} = -\frac{4\pi}{3} \rho r^2 e^{-\frac{(ra)^2}{b}} \).

Thus we have

\[ E_{GR} = -\int_0^R \frac{4\pi}{3} \rho r^2 e^{-\frac{(ra)^2}{b}} \, dr = -3 \left( \frac{4\pi}{3} \right) \rho \frac{R^3}{3} \int_0^R e^{-\frac{(ra)^2}{b}} \, dr \]

Let \( x = ra \) so \( r = ax \) \& \( dr = a \, dx \). Then

\[ E_{GR} = -3 \left( \frac{4\pi}{3} \right) \rho \frac{R^3}{3} a^5 \int_0^R x^3 e^{-x^2} \, dx \]

But if \( a \ll R, \frac{R}{a} \gg 1 \) (e.g. if \( a = \frac{1}{2} R, \frac{R}{a} = 5 \) so \( e^{-x^2} \rightarrow e^{-25} \) the value of the integral is nearly unchanged if we let the upper limit \( \rightarrow \infty \). Then \( \int_0^\infty x^3 e^{-x^2} \, dx = \frac{3\sqrt{\pi}}{8} \) and we find

\[ E_{GR} = -3 \left( \frac{4\pi}{3} \right) \rho \frac{R^3}{3} a^5 \frac{3\sqrt{\pi}}{8} \frac{\sqrt{\pi}}{3} \]

(eq. 5.32) so that \( \left( \frac{4\pi}{3} \right) \rho \frac{R^3}{3} a^5 \frac{3\sqrt{\pi}}{8} \frac{\sqrt{\pi}}{3} = \frac{1}{6 \, a^2} M^2 \). Combine the two and

\[ E_{GR} = -3 G \frac{a^5}{6 \, a^2} M^2 \frac{3\sqrt{\pi}}{8} \frac{\sqrt{\pi}}{3} = -\frac{3\sqrt{\pi}}{16} G M^2 \]

to put this into the familiar \( GM^2/r \) form for the potential, we write \( \frac{1}{a} = \left( \frac{R}{a} \right) \frac{1}{R} \) to obtain

\[ E_{GR} = -\frac{3\sqrt{\pi}}{16} \left( \frac{R}{a} \right) \frac{GM^2}{R} = -0.3323 \left( \frac{R}{a} \right) \frac{GM^2}{R} \approx \frac{1}{3} \left( \frac{R}{a} \right) \frac{GM^2}{R} \]

for \( (R/a) = 5 \), then \( E_{GR} = -\frac{5}{3} \frac{GM^2}{R} = -\frac{5}{3} \frac{GM^2}{R} \) with \( f = \frac{5}{3} \)

But for a polytrope of index \( n \), \( f = \frac{3}{5 - n} \) (class notes)

Thus \( n = 5 - \frac{3}{f} \) so \( n = \frac{16}{5} = 3.2 \).
We are considering the gas and radiation at the center of a star. \( \beta \) is \( P_g / P_c \), where \( P_c \) is the total pressure. Then \( (1-\beta) = P_r / P_c \). The gas pressure is just

\[
P_g = \frac{n k T_c}{m} \frac{4}{5} \frac{k T_c}{P_c}
\]

\( \beta = \frac{k}{m} \frac{n e}{P_c} T_c \)

The radiation pressure is \( P_r = \frac{1}{3} a T_c^4 \) where \( a \) is the radiation constant \( a = 7.566 \times 10^{-15} \) erg cm\(^{-3}\) K\(^{-4}\). Thus we have

\[
(l-\beta) = \frac{a}{3} \frac{T_c^4}{P_c}
\]

Following p.237 of Phillips, we evaluate

\[
\frac{(l-\beta)}{\beta^4} = \left( \frac{a}{3} \frac{T_c^4}{P_c} \right) \left( \frac{k}{m} \frac{n e}{P_c} T_c \right) = \frac{a}{3} \left( \frac{k}{m} \right) \frac{n e}{P_c} T_c^4
\]

Now we have \( P_c < \left( \frac{\pi}{6} \right)^{1/3} GM^2 \frac{m}{T_c^4} \), so \( P_c^3 < \frac{\pi}{6} GM^2 \frac{m}{T_c^4} \)

Put this into the previous expression to get

\[
\frac{(l-\beta)}{\beta^4} < \frac{\pi}{18} \frac{n e}{m} \frac{a}{k} \frac{GM^2}{T_c^4}
\]

Let's write \( \bar{m} \) as \( \mu \frac{m}{H} \), where \( \mu \) is the mean molecular weight in units of the hydrogen mass. Then

\[
\frac{l-\beta}{\beta^4} < C_1 \mu \frac{(M_{\odot})^2}{(M_{\odot})^2}
\]

where \( C_1 = \frac{\pi}{18} \frac{n e}{m} \frac{a}{k} \frac{GM^2}{T_c^4} \)

Putting in the constants, we have \( C_1 = 0.033515 \)

For a mix of hydrogen and helium \( \mu \simeq 4/(3+5X) \) so for the original solar composition, \( X = 0.7, Y = 0.3 \), we get \( \mu \simeq 0.6 \)

So finally

\[
\frac{l-\beta}{\beta^4} < 0.00434 \left( \frac{M_{\odot}}{M_{\odot}} \right)^2
\]

For \( M = 4 M_{\odot} \), \( \frac{l-\beta}{\beta^4} < 0.0635 \)

And for \( 40 M_{\odot} \), \( \frac{l-\beta}{\beta^4} < 6.95 \)
cont. Now, if we were to plot \((1 - \beta)/\beta^4\) over the range of \(0 \leq \beta \leq 1\), the curve would look like this:

\[
\begin{array}{c}
\frac{(1 - \beta)}{\beta^4} \\
\beta \rightarrow 1
\end{array}
\]

So the higher \(\beta\), the smaller \((1 - \beta)/\beta^4\).

That means the inequality will be reversed for \(\beta\).

By plugging some values we find that \((1 - \beta)/\beta^4 = 0.0659\) when \(\beta = 0.944655\) and also \((1 - \beta)/\beta^4 = 6.75\) for \(\beta = 0.514187\). Thus our resulting limits are

\[
M = 4 M_\odot, \quad \beta > 0.944655 \Rightarrow \frac{P_{\text{rad}}}{P_c} = (1 - \beta) < 0.055
\]

\[
M = 40 M_\odot, \quad \beta > 0.514187 \Rightarrow \frac{P_{\text{rad}}}{P_c} = (1 - \beta) < 0.4858
\]

Radiation can't have more than a 5% contribution at the center of a 4M_\odot star, but can have up to 50% of the pressure for a 40 M_\odot star.
The density of atomic matter is of the order
\[ \rho_{\text{at}} = \frac{m_H}{\alpha_B^3} = m_H \alpha^3 E M \frac{m_e^3 c^3}{h^3} = m_H \alpha^3 \frac{m_e^3 c^3}{h^3} (2\pi)^3 \]

(This would work better if the \( h \) were \( h \sqrt{2} \) we didn't get the \( (2\pi)^3 \) !)

From equation 6.4, the central density of a (non-relativistic) degenerate object is given by
\[ \rho_c = \frac{3.1}{Y_e^3} \left( \frac{M_e^3}{m_H} \right)^{3/2} \frac{m^3 c^3}{h^3} \]

If we set \( \rho_{\text{at}} = \rho_c \) we get
\[ \frac{3.1}{Y_e^3} \frac{m^2}{m_H} \frac{\alpha^3}{\alpha_g} \frac{m_e^3 c^3}{h^3} = \frac{m_H^3}{m_H^3} \frac{m_e^3 c^3}{h^3} \frac{(2\pi)^3}{(2\pi)^3} \]
\[ m^2 = m_H^2 \frac{Y_e^3}{3.1} \frac{\alpha^3}{\alpha_g} \frac{m^3 c^3}{h^3} \]
\[ m = \frac{Y_e^{5/2} (2\pi)^{3/2}}{(3.1)^{1/2}} \left( \frac{\alpha_{EM}}{\alpha_g} \right)^{3/2} m_H \]

We would get the result in Phillips if \( \left[ \frac{Y_e^{5/2} (2\pi)^{3/2}}{3.1} \right]^{1/2} \approx 1 \).

What's \( Y_e \)? The number of electrons per nucleon. Phillips gives the expression \( Y_e = (1+x)/2 \). For a white dwarf it would be \( 1/2 \), but we are talking about big planets. For a solar composition, \( x \approx 0.7 \) so \( Y_e \approx 0.85 \). Then
\[ \left[ \frac{0.85^{5/2} (2\pi)^{3/2}}{3.1} \right]^{1/2} = 5.96 \]
so that \( m \approx 6 \left( \frac{\alpha_{EM}}{\alpha_g} \right)^{3/2} m_H \)
\( \left( \frac{\alpha_{EM}}{\alpha_g} \right)^{3/2} m_H = \frac{8}{3} \alpha_{EM} \frac{1}{1.85} M_\odot = 0.00115 M_\odot \)
So, I get \( M_p \approx 0.007 M_\odot = 7.3 M_{\text{Jupiter}} \).

As I said above, if \( h = h \), we wouldn't have that \( (2\pi)^{3/2} \), and we would be closer to Phillips' result.