Prototyping Problems in Astrophysics with J

J. Patrick Harrington
University of Maryland

Historically, many astrophysical problems demand heavy computation.
For example:

1) stellar structure and evolution, especially advanced phases
2) numerical magnetohydrodynamics: transfer of angular momentum in ionized accretion disks and formation of jets
3) numerical general relativity: only recently has the merging of two black holes been solved.
4) big n-body problems: the time evolution of millions to billions of point masses, e.g. collision of two galaxies.

J is not suitable for such problems -- the codes are generally written in C (or even Fortran), typically in parallel versions that run under MPI.

But there are many smaller problems, especially in developmental stages, where I've found J quite useful. I can write and modify code more rapidly than in other languages.

Of course, J is quite handy as a general purpose "scratch pad", where you have defined all the physical & mathematical constants you generally use -- for example I would load a script like:

```
AU=: 1.4959787066e13
c=: 2.99792458e10      NB. exact
e=: 4.8032042e_10
h=: 6.62606876e_27
h_bar=: 1.05457159642e_27
eV=: 1.60217646e_12
G=: 6.6731e_8
k=: 1.3806503e_16
pc=: 3.0856776e18
LYR=: 9.460730472e17
yr=: 31556926
sig=: 5.67040e_5
L_sun=: 3.845e33
M_sun=: 1.9891e33
R_sun=: 6.95508e10
T_sun=: 5779.16
m_e=: 9.10938188e_28
m_H=: 1.6735325e_24
R_H=: 109677.583406
R_inf=: 109737.3156855
amu=: 1.6605402e_24
m_p=: 1.67262158e_24
m_n=: 1.67492716e_24
N_A=: 6.02214199e23
```

J is nice for such calculations since you often want to evaluate an equation with one or more variables taking a range of values to generate a list or a table. That's a handy, but rather trivial application.

To illustrate some more substantial calculations using J, I will focus on one set of problems where we are at the exploratory stage. We are not trying to write an efficient bullet-proof code, we are just trying out various approaches to the problems. These examples have to do with what is called "radiative transfer": the generation, scattering and absorption of light by the gaseous atmospheres of stars and planets.

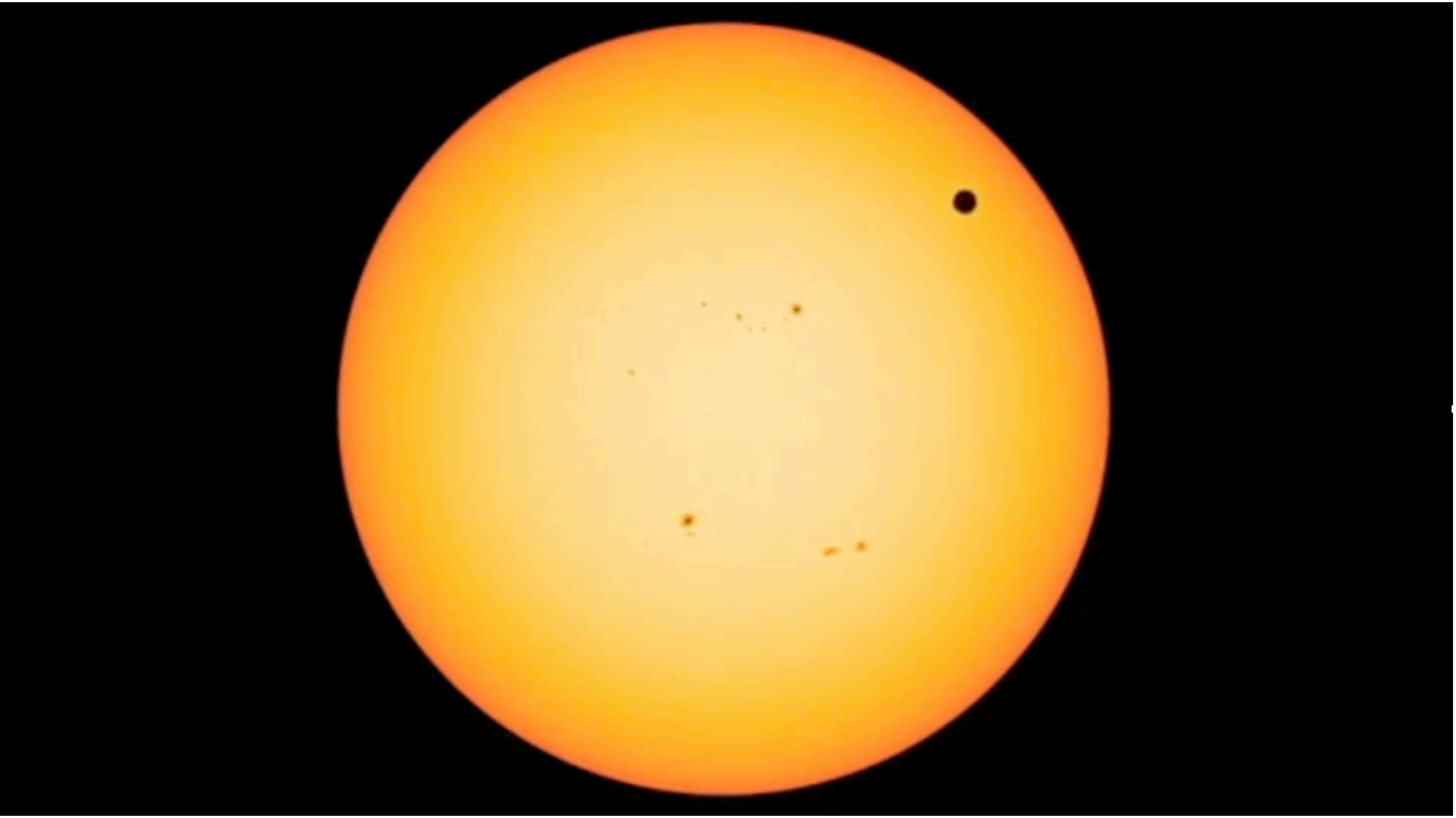
In particular, I'm interested in the POLARIZATION of the scattered light. When light is scattered by molecules or small particles, it becomes partially polarized. One consequence of this is that the atmospheres of stars which have scattering agents (free electrons in hot stars, hydrogen molecules in cool stars) will polarize the light emerging from them. We don't think of starlight as polarized, but that is because the spherical symmetry of the stellar disk results in equal polarization in all directions and hence complete cancellation.

So if stars show no observable polarization because of this symmetry, why do I care? Because some conditions may break the symmetry.

One of the most interesting topics of research today is the effort to discover and characterize planets about other stars: exoplanets. The first exoplanets were discovered in 1995, though astronomers had searched for decades before. They are surely very common but really hard to identify. We have now identified over 600 exoplanets.

One of the most fruitful methods for finding exoplanets is by observing the slight decrease in light caused by the transit of a planet across the star as it passes between us and the star. This is the method used by the NASA Kepler Spacecraft, which constantly stares at around 145,000 stars in a patch of sky in Cygnus.

The light decrease is small, equal to $(R\_planet/R\_star)^2$. For Jupiter, that's only 0.01 -- for Venus and the Sun, 7.6e_5. Worse, the chance that we are close enough to the plane of the orbit of the planet is about 1 in 160 for Venus and only 1 in 1000 for a planet the distance of Jupiter. That's why we must monitor huge numbers of stars.

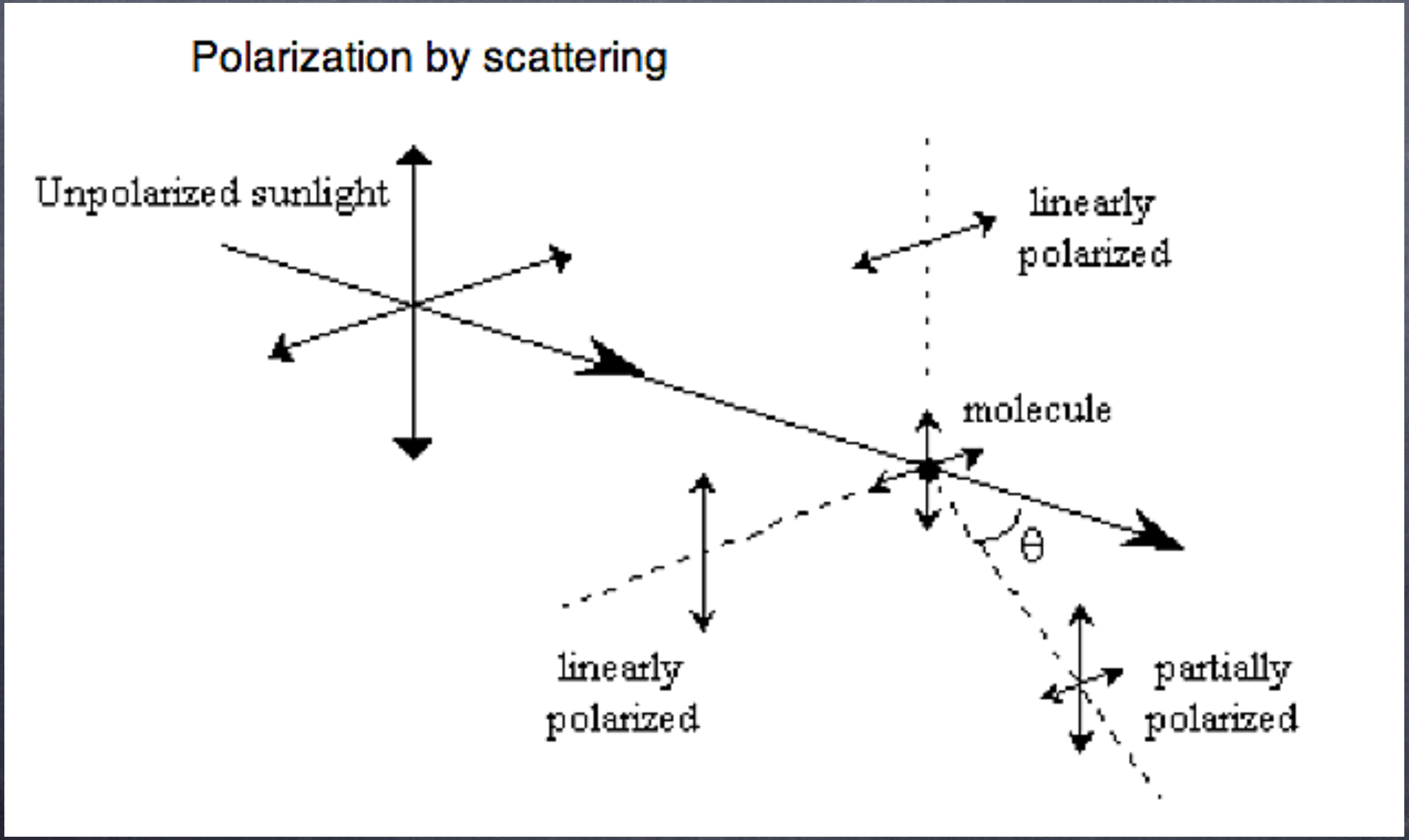


2012 Venus Transit

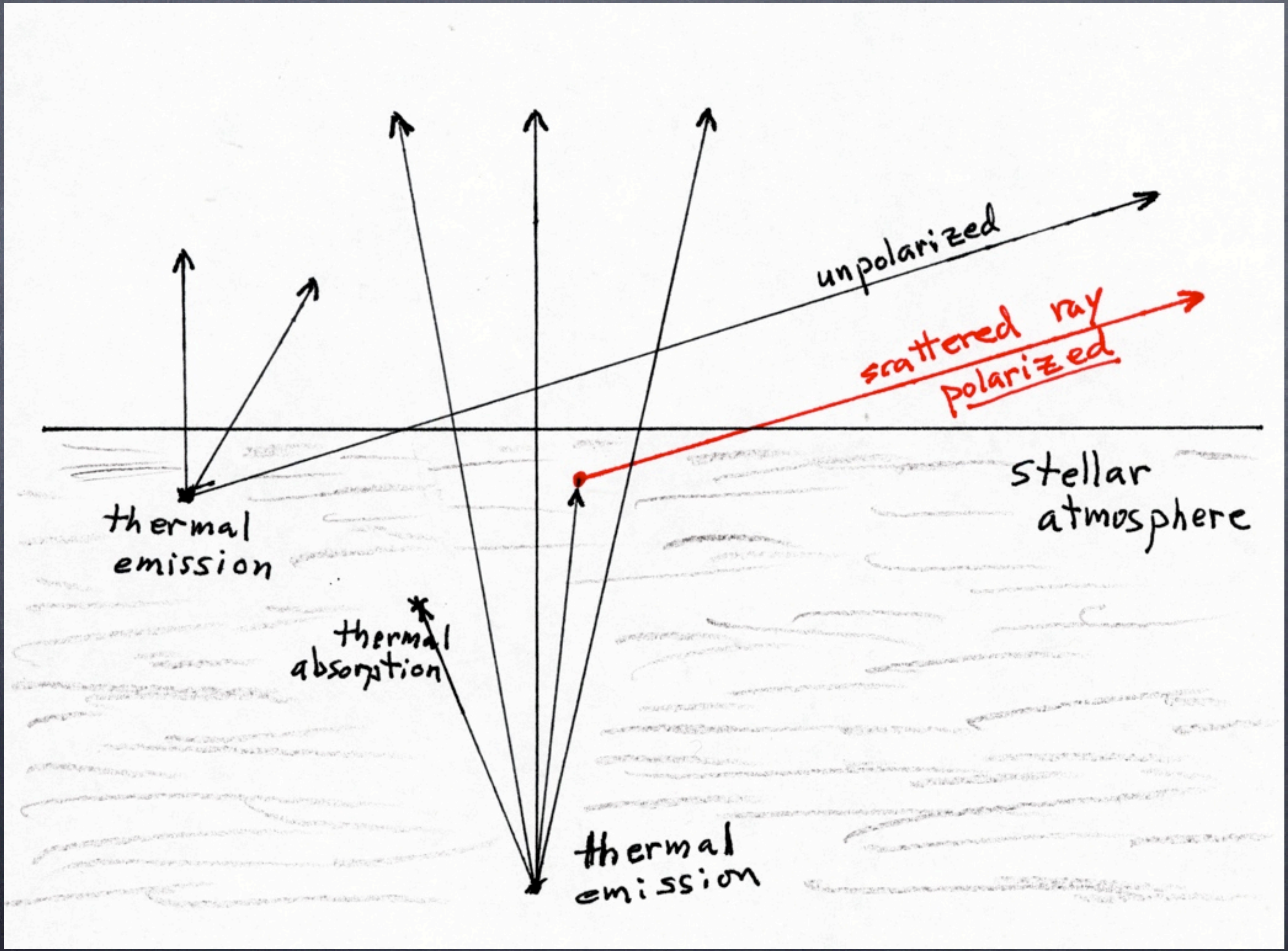
A rare transit in our solar system.
Venus crosses the solar disk.

So what's the connection between exoplanet transits and polarization? Let's look at stellar polarization.

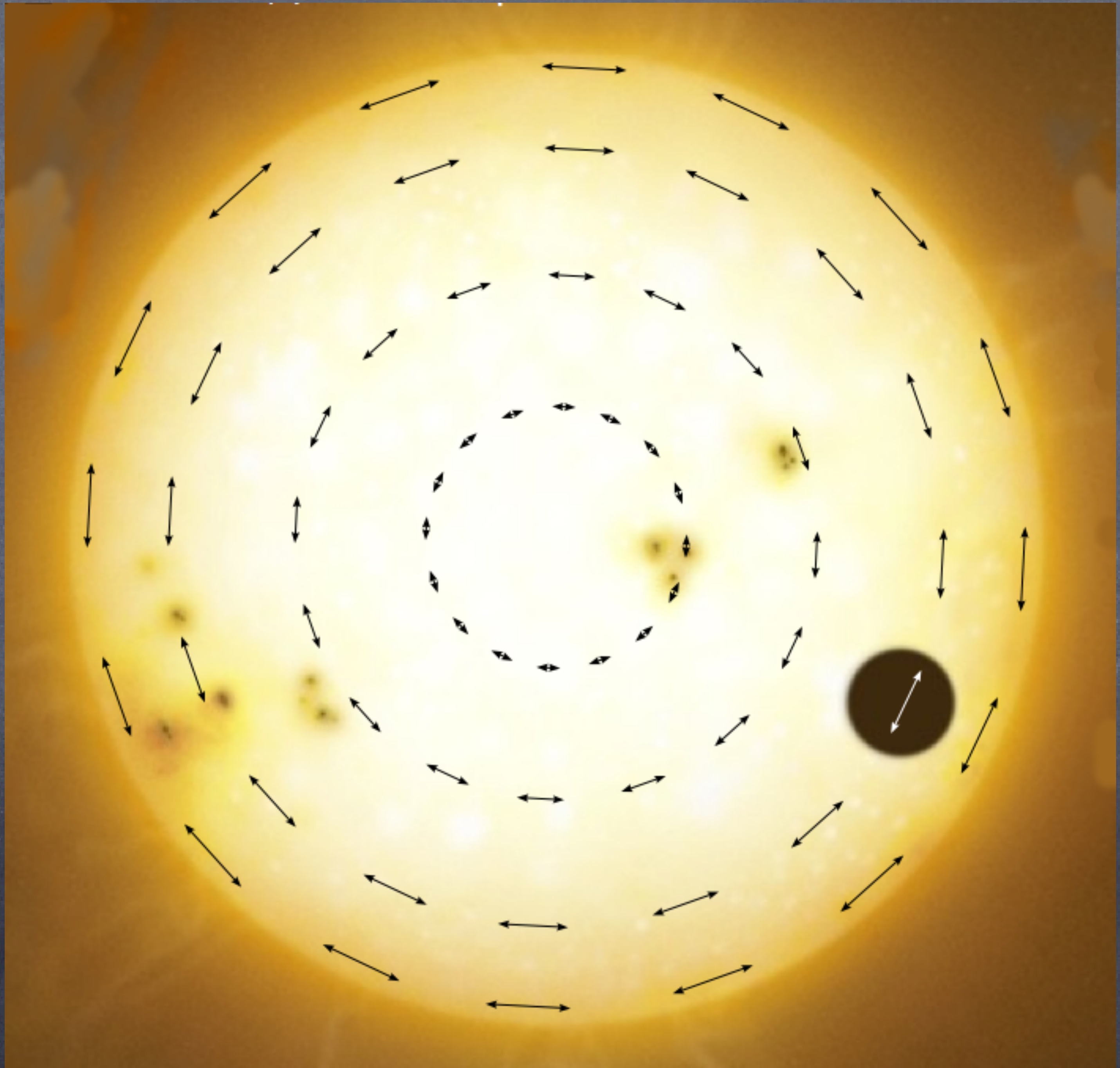
The simplest type of scattering law, which applies to free electrons and to gas molecules, is Rayleigh scattering. It produces 100% polarization when the scattering angle is 90 degrees. The angle of polarization is perpendicular to the plane of scattering:



The polarization of light emerging from a stellar atmosphere is greatest for rays nearly tangent to the surface. The degree of polarization depends on the ratio of scattering to absorption in the upper layers of the star's atmosphere. The plane of the polarization is parallel to the surface of the atmosphere.



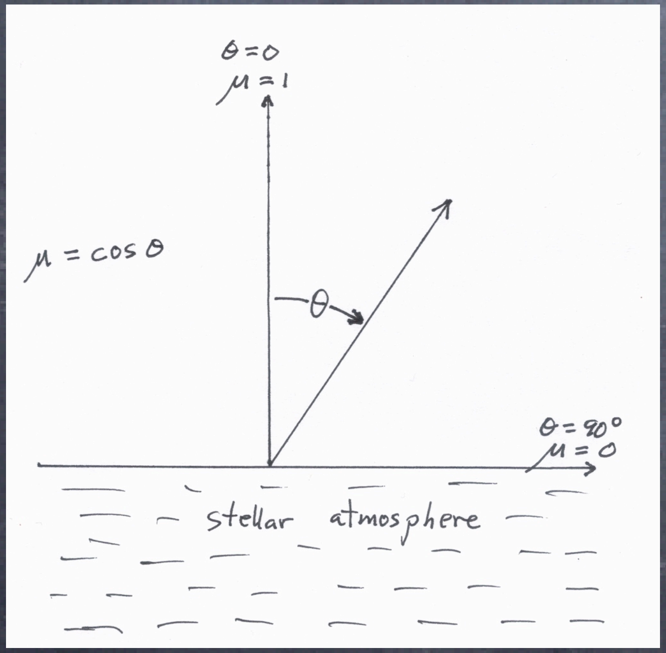
This figure shows the magnitude and orientation of the polarization from various points on the star's surface. We see that averaged over the entire star, the net polarization is zero. But if something breaks the symmetry, then the cancellation is not exact and we may observe a net polarization. For example, the transit of a planet across the stellar disk:



The background image is an artists concept of a transiting exoplanet. Overlaid are polarization vectors. The black vectors will be observed while the white vector is blocked by the planet.

To find the fractional polarization we need a model atmosphere:
A model atmosphere is a table of physical parameters such as the temperature, density, opacity, scattering, etc. as a function of depth. Such models exist for many types of stars. The models also give the emergent radiation, but but the models never include polarization. So I have written some J code to extract it from the models.

The emergent intensity as a function of the angle parameter μ (μ = cos ϑ), is obtained from the source function, s(τ):



$$I(0,\mu) = \int_0^\infty s(\tau)\, e^{-\tau/\mu} \frac{d\tau}{\mu}$$

Without scattering, the source function is just the Planck function B(τ), but with scattering, we must solve this integral equation:

$$s(\tau) \;=\; (1-\lambda)\,\Lambda_\tau(s) \;+\; \lambda\,B(\tau)$$
$$\lambda = \;\mathrm{abs}/(\mathrm{abs}+\mathrm{scat})$$

Here, the s(τ) is operated on by the Λ-transform, defined as

$$\Lambda_\tau\{f(t)\} \;=\; \frac{1}{2}\int_0^\infty f(t)\;E_1(|t-\tau|)dt$$

The kernel of the Λ-transform is the 1st exponential integral function.
The exponential integral, E_n, of order n, is defined as:

$$E_n(a)=\int_1^\infty x^{-n}e^{-ax}dx.$$

All this is for isotropic scattering without polarization.

If we include polarization, we find that we must introduce a 2nd equation and a 2nd function, p(τ). The integral of p(τ) over depth gives us the polarization of the emergent radiation.

$$s_\nu(\tau_\nu) = (1-\lambda_\nu)\left[\Lambda_{\tau_\nu}(s_\nu) + \frac{1}{3}M_{\tau_\nu}(p_\nu)\right] + \lambda_\nu B_\nu(\tau_\nu)$$

$$p_\nu(\tau_\nu) = \frac{3}{8}(1-\lambda_\nu)\ [M_{\tau_\nu}(s_\nu) + N_{\tau_\nu}(p_\nu)]$$

Here, Λ_τ is the familiar Λ-operator, (familiar to astronomers!)

$$\Lambda_\tau\{f(t)\} = \frac{1}{2}\int_0^\infty f(t)\ E_1(|t-\tau|)\ dt \quad .$$

In addition, we have to introduce two new transforms, which I have called the M-transform and the N-transform:

$$M_\tau\{f(t)\} = \int_0^\infty f(t)\left[\frac{1}{2}E_1(|t-\tau|) - \frac{3}{2}E_3(|t-\tau|)\right]\ dt$$

$$N_\tau\{f(t)\} = \int_0^\infty f(t)\left[\frac{5}{3}E_1(|t-\tau|) - 4E_3(|t-\tau|) + 3E_5(|t-\tau|)\right]\ dt$$

To solve these equations, we may compute a matrix Λ_{ij} which, when it multiplies a vector of discrete function values s_i, yields a good approximation of the integral transform $\Lambda(s)$ of the function $s(\tau)$. I do this by assuming $s(\tau)$ and $p(\tau)$ can be well represented by cubic splines. We also do this for the M and N-transforms. So now we can write the equations as a linear system to be solved for the unknown vectors s_i and p_i. This is the resulting system:

$$\begin{bmatrix} (1-\lambda_i)\Lambda_{ij} - I_{ij} & \frac{1}{3}M_{ij} \\ \frac{3}{8}(1-\lambda_i)M_{ij} & \frac{3}{8}(1-\lambda_i)N_{ij} - I_{ij} \end{bmatrix} \begin{bmatrix} s_i \\ p_i \end{bmatrix} = \begin{bmatrix} \lambda_i B_i \\ 0 \end{bmatrix}$$

In order to find the elements of Λ_{ij}, M_{ij} and N_{ij}, we must evaluate a large number of exponential integral functions of orders 1-5. The J code I use for this is more than fast enough. One million calls takes 14 sec:

```
   a=. 5*rand 1e6
   ts'z=. 1 EIn a'
13.6802 3.12481e8
   ts'z=. 4 EIn a'
13.863 3.19821e8
```

Here's the code for the EIn verb:

```
NB. Exponential integral functions: n EIn x
NB. x may be a vector, n should be a single value.
EIn=: dyad : '((y>1)*x&EnCF y)+((y<:1)*x&EnS y)'

NB. For x<=1, we use the series expansion.
NB. n EnS x  (should be accurate to machine limit)
NB. order "n" must be single value, x may be a vector
gamma=: 0.577215664901532860606512   NB. Euler's constant
EnS=: dyad define
psi=. - gamma- +/%>: i. nm1=._1+ n=.x
b=. (psi- ^.y)* (!nm1)%~ nm1^~ -y
a=. (!k-1)* n-k=. >:i. nm1
a=. +/ a%~ (i. nm1)^~/ -y
d=. k* !kk=. n+ <: k=. >: i. 16
r=. a+ b- +/ d%~ kk^~/ -y
if. n=1 do.   NB. remove E_1(0)= infinity
  r=. 0 (I. y<:0)} ,((1,(#r))$ r)
end.
r
)

NB. For x>1 we use a continued fraction expansion.
NB. Usage n EnCF x (error of a few times 10^-16).
NB. Follows code on p 217 of Numerical Recipes
EnCF=: dyad define
nm1=. _1+ n=. x   NB. nm1 = n-1
b=. n+y
c=. 1e300
h=. d=. %b
i=. 0
while. (i=.>:i)<100 do.
  a=. -i*(i+nm1)
  d=. %(a*d)+ b=. b+2
  c=. b+ a%c
  h=. h* del=. c*d
end.
h* ^-y
)
```

Code for x <= 1

Code for x > 1

I evaluate the expressions for both x>1 & x<1 and zero out one of them. The same number of terms is used for all values of the argument.

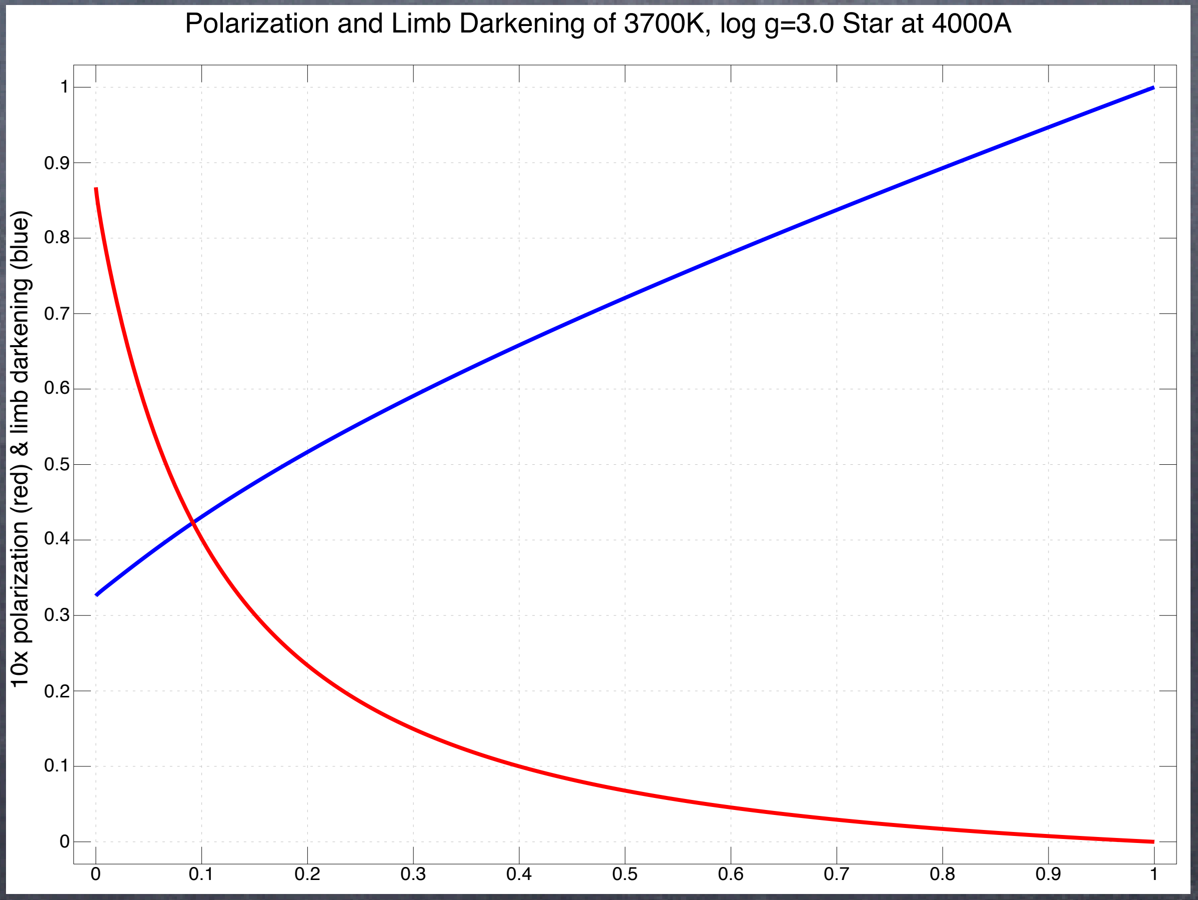
Once we have the matrix representations of the transforms, solving the linear system is quick (80 x 80 matrices work fine). The fractional polarization is just pol(μ)=Q(μ)/I(μ) for any angle. Here Q(μ) is the so-called Stokes parameter that we find from p(τ).

$$Q(0,\mu) \;=\; \int_0^\infty (1-\mu^2)p(\tau)\; e^{-\tau/\mu}\frac{d\tau}{\mu}$$

With this code I can quickly generate the polarization and "limb darkening" (the variation of I with ϑ) for any of the 50,000 or so model atmospheres available on various websites. E.g.:

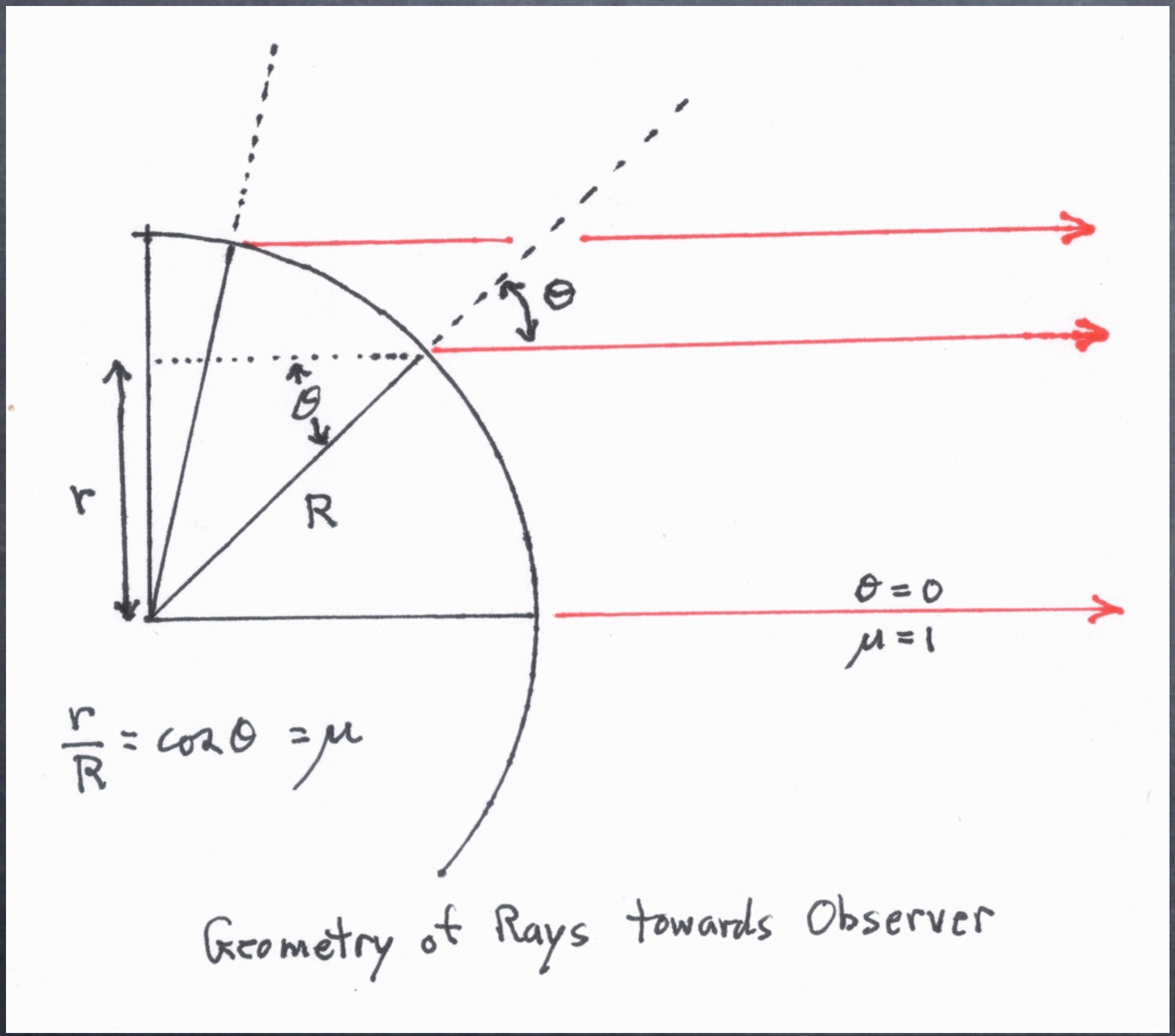
red = fractional polarization

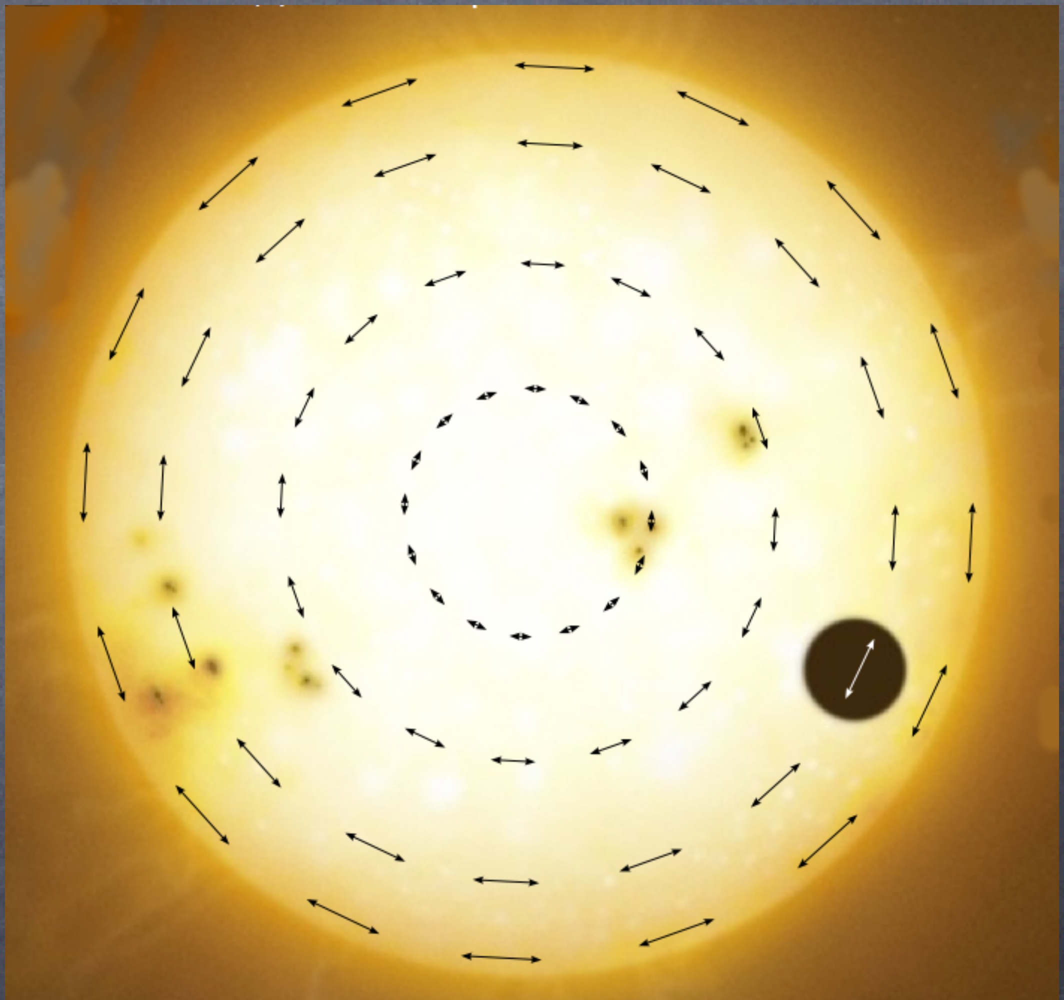
blue = intensity I(μ)



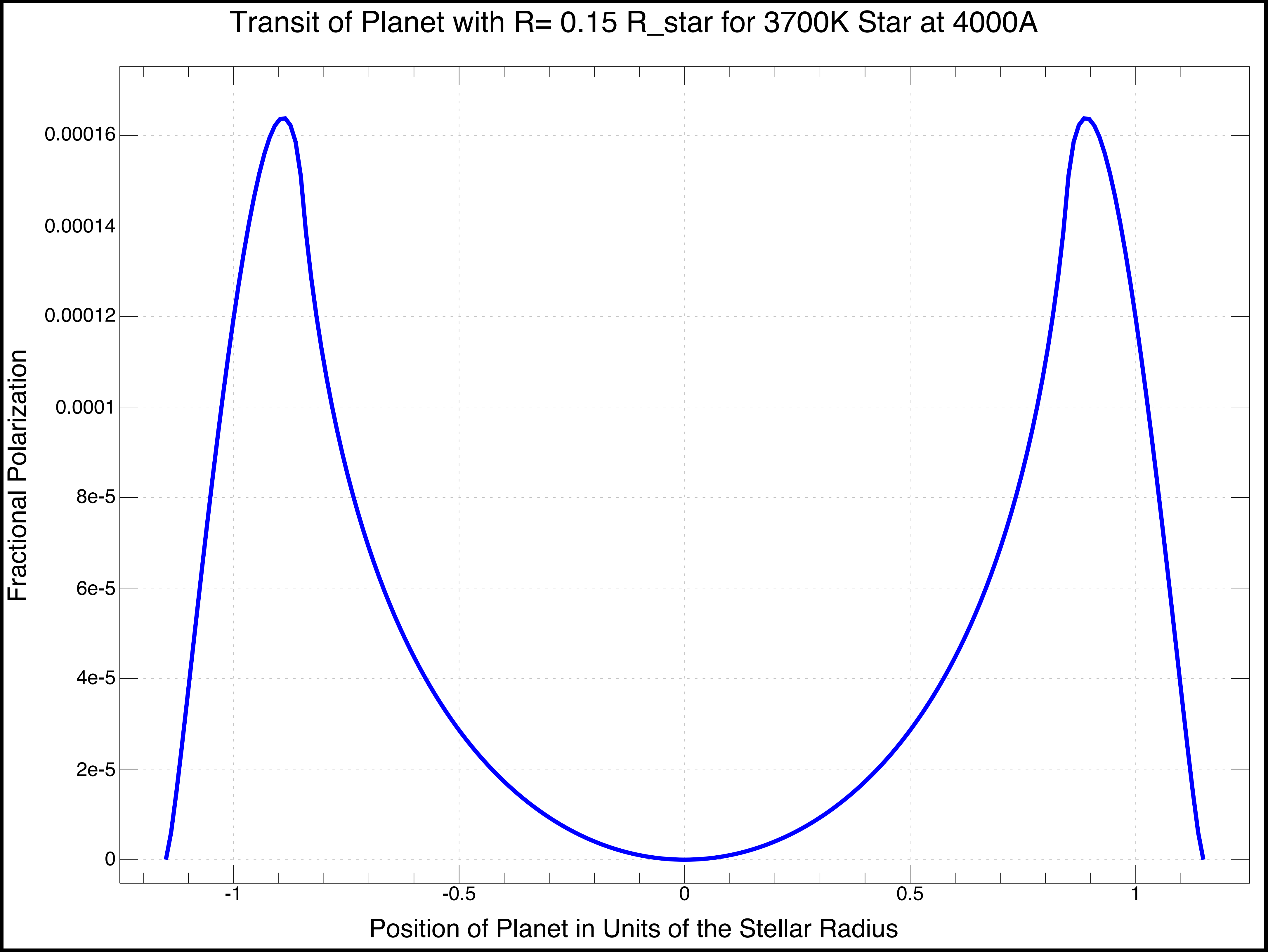
edge μ = cos ϑ --> center

So now let's go back to our star with the transiting exoplanet. We now have the polarization from each point on the stelar disk (the lengths of the little arrows), so it is a simple matter to find the net polarization from the star when a planet blocks part of the stellar surface. (Actually, we only need to integrate over the area blocked by the exoplanet and reverse the sign.)

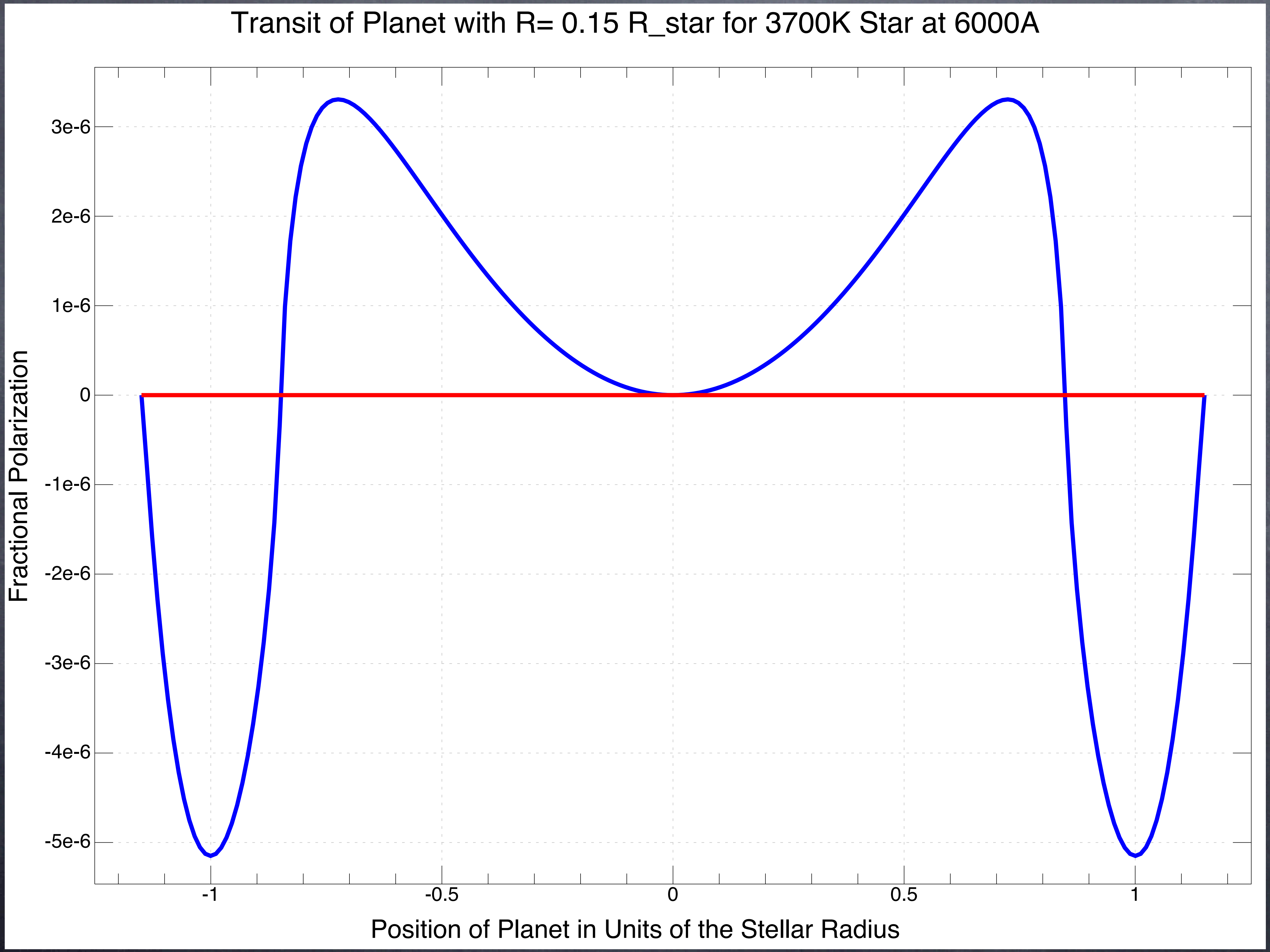



The background image is an artists concept of a transiting exoplanet. Overlaid are polarization vectors. The black vectors will be observed while the white vector is blocked by the planet.

Here are some results -- polarization peaks when planet is near the edge of the star. It is quite small, with the maximum around 0.016 %. Such small polarization fractions are not observable now, but may be in the future.



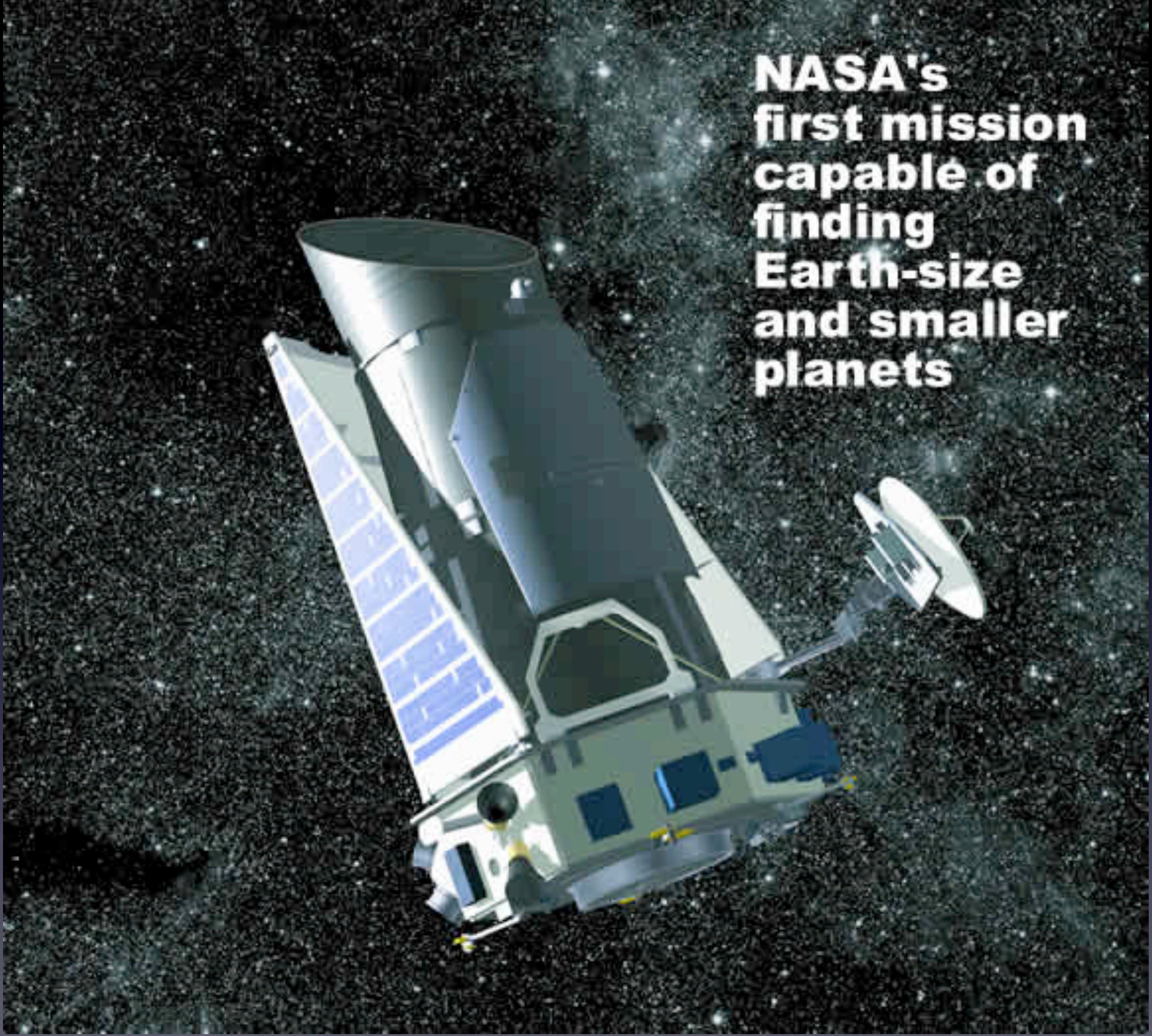
At longer wavelengths, the polarization is reduced but the change of sign means the polarization angle flips by 90 degrees.



So far we have been considering the light from the star only.

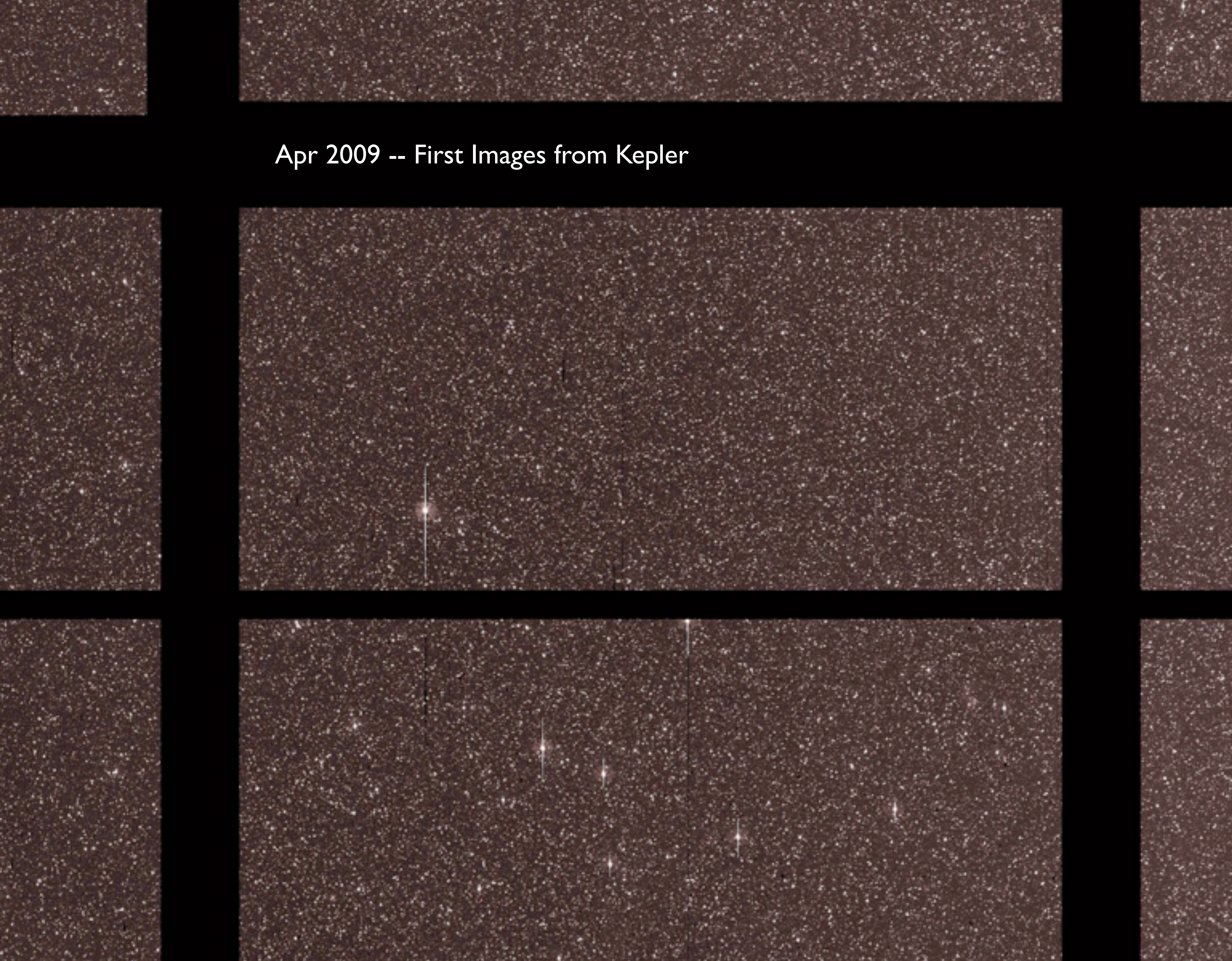
But for large planets close to their stars, the so-called "hot Jupiters", we may actually observe starlight scattered by the planet's atmosphere.

Let's see what sort of effects we are talking about and then consider the part that polarization might play.

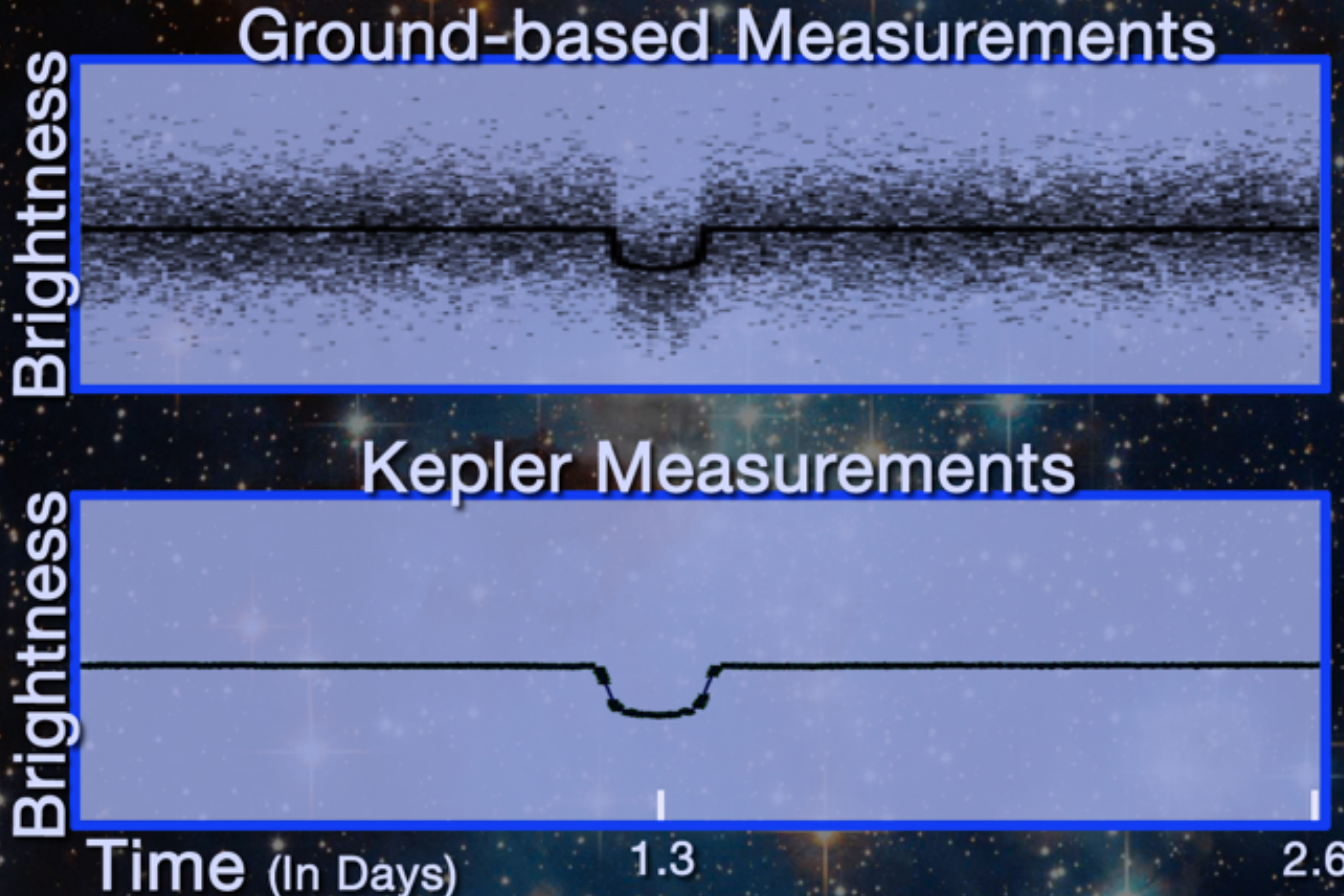


KEPLER (Launched 7 March 2009)

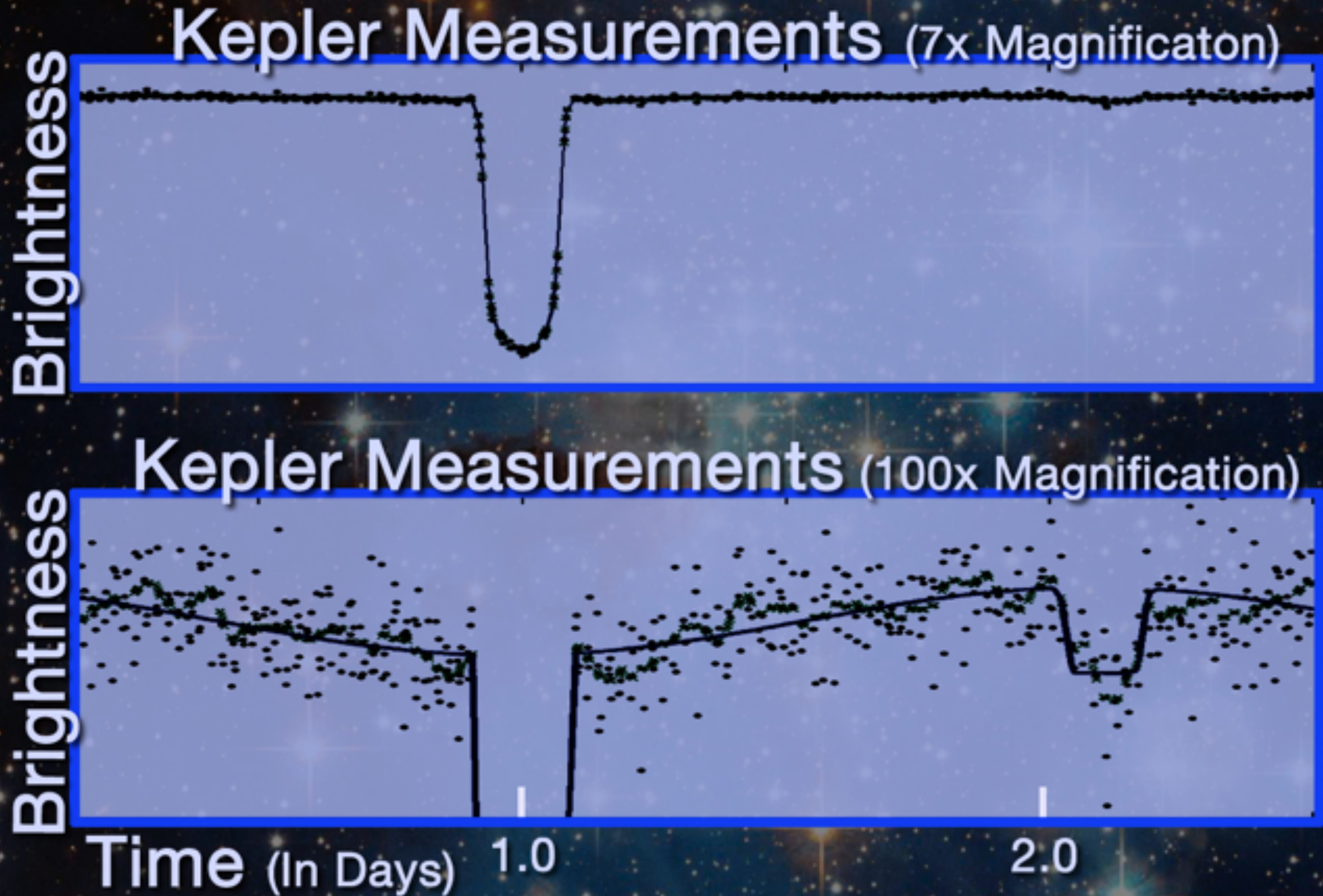
Apr 2009 -- First Images from Kepler

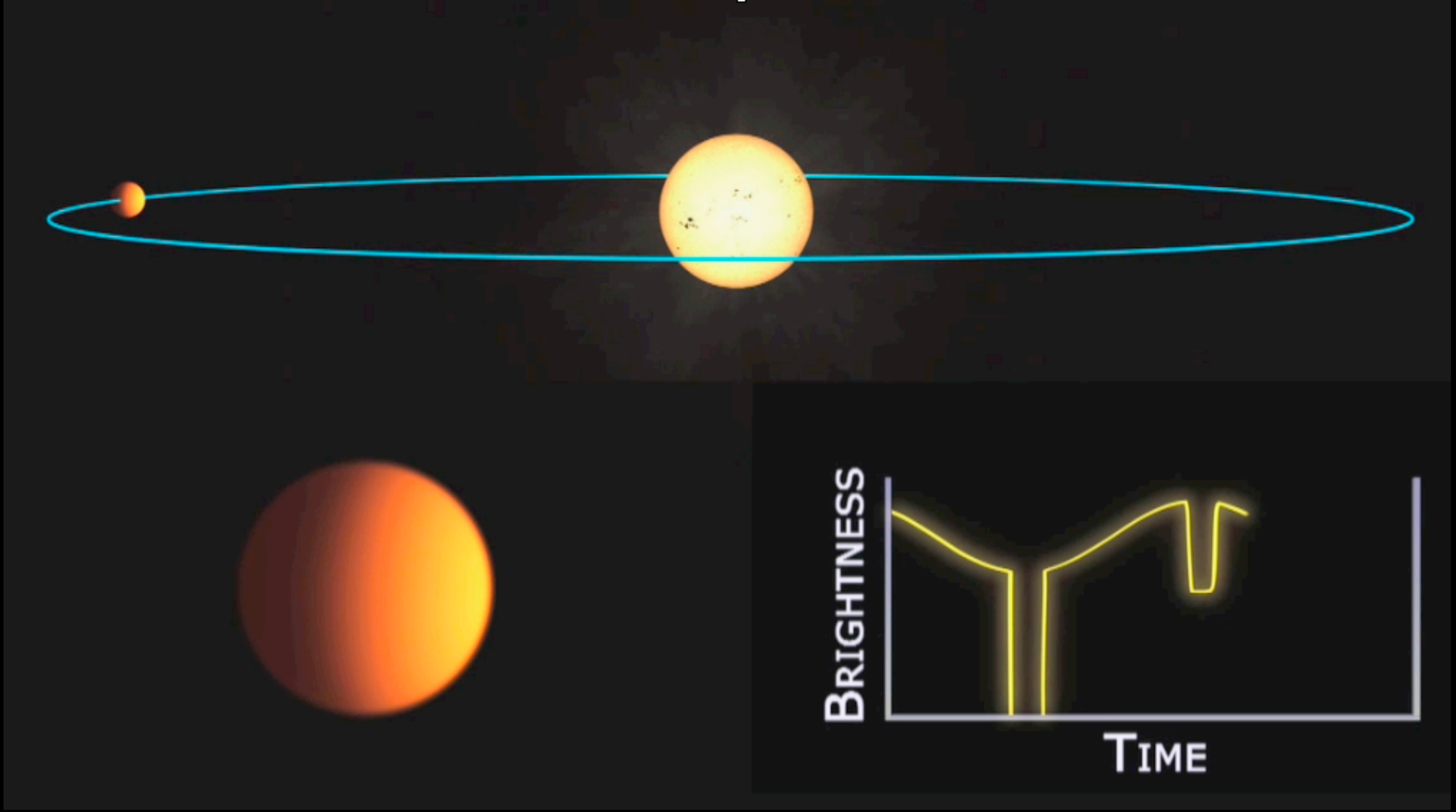


HAT-P-7 Light Curves

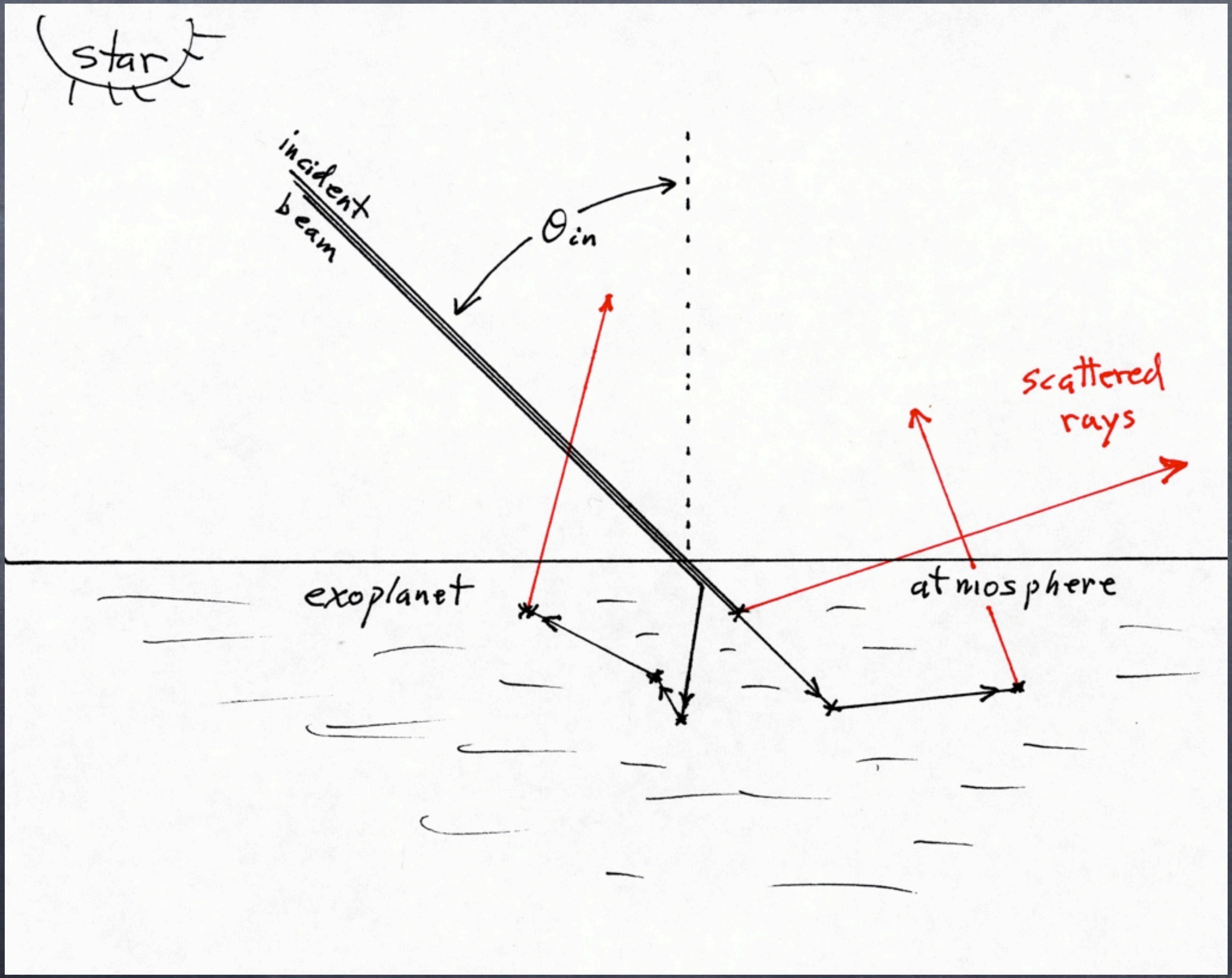


HAT-P-7 Light Curves




BRIGHTNESS
TIME

To evaluate these effects, we need to solve the problem of the scattering of a beam of radiation incident on a planetary atmosphere. Since we don't have a good direct way to solve this problem, we have used a Monte Carlo approach, where we follow many bundles of photons as they scatter around and eventually emerge.

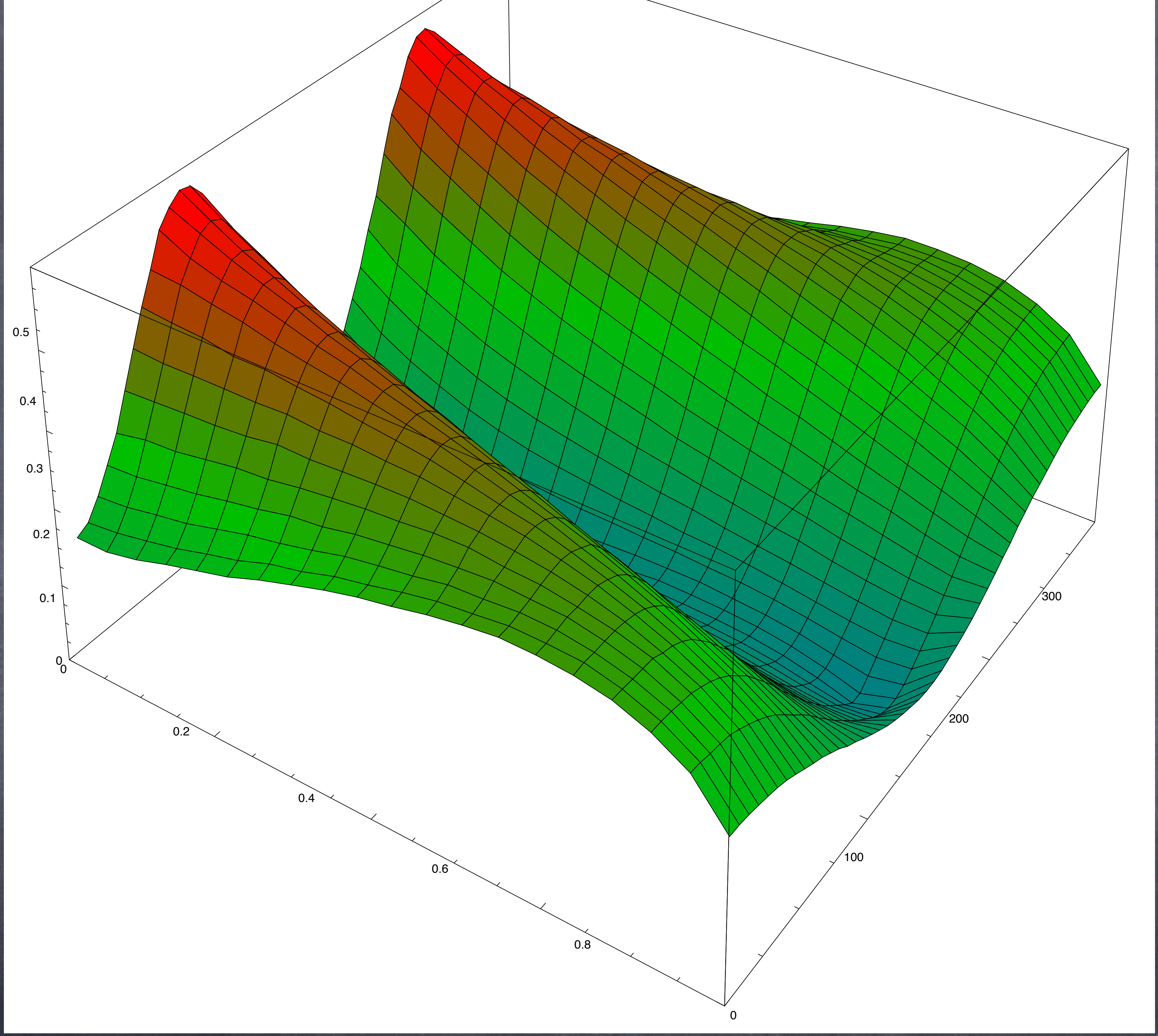


Each “photon” is characterized by 4 Stokes parameters (which define its intensity and polarization state), by 3 direction cosines giving the direction of travel after scattering, and by the coordinates of where it is scattered. If we are following N photons, then we have an N x 10 matrix, where N is as large as possible without causing page faults. We have used a value of N = 300,000 for many runs. At each scattering, we compute the probability that the photon will escape along the new trajectory. If the probability is “f”, the we record the intensity f*I as escaping, and continue to scatter an intensity (1-f)*I, so the photons never escape totally, but decline in intensity with each scattering. For thick atmospheres, we may follow as many as 60 scatterings. We repeat this hundreds of times so that the emergent intensity pattern is sufficiently smooth. Each group of N may take, say, 3 minutes, and some of the runs may take days to get the desired statistics.

The escaping photons are characterized by their intensity I, their degree and angle of polarization, and the two angles (altitude and azimuth) at which the emerge relative to the surface of the atmosphere and the azimuth of the incident beam. We have to bin the N photons after each scattering into altitude and azimuth bins, and sum the intensities, etc. in each bin. This would be a storage/time bottleneck if we didn’t have the J verb key:

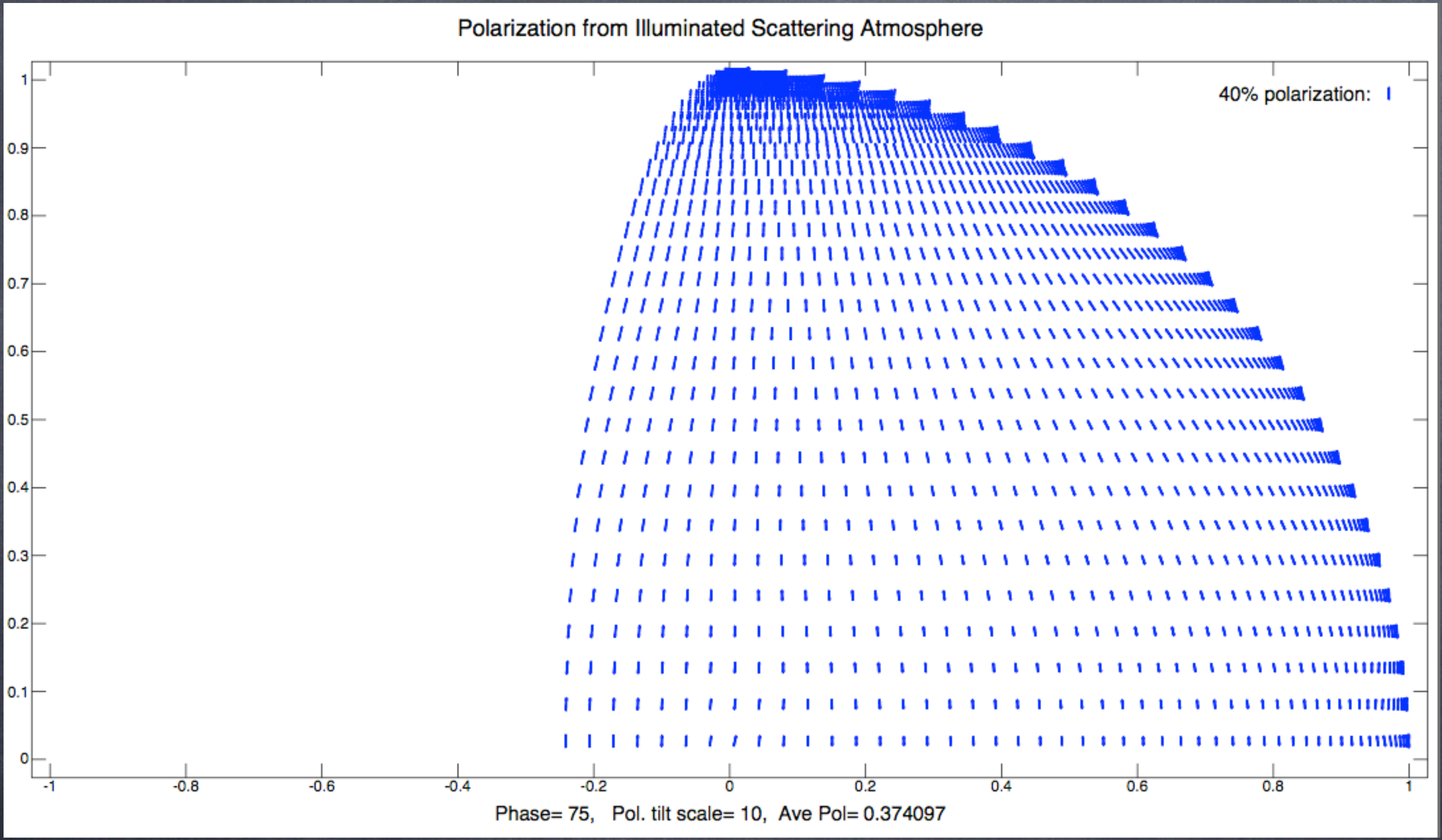
(indices of photons altitude & azimuth bin) +//. (Stokes parameters of photon).

For a complete solution, we have to construct a 2-dimensional table of scattered radiation for all relevant angles of incident radiation ϑ_{in}. Then we are finally able to see how each point on the exoplanet surface scatters light toward the distant observer.



Fractional polarization for beam entering a thick scattering atmosphere at a 45 degree angle. (Altitude as $\mu = \cos\vartheta$)

We can use these results to see the polarization of the scattered radiation over the face of the exoplanet:



We can then integrate over the exoplanet surface to get the total intensity of the scattered light and it's degree of polarization.

If we can detect such polarization, what can we learn?
(1) the direction of the polarization gives us the orientation of the orbit.
(2) variation in the fraction of polarization with wavelength will provide information on the nature of the scattering atmosphere (clouds?).

Perhaps most interesting is the case of an orbit that we see face on. In that case there will be no transits, and no radial velocity variations. No way to detect the planet.

But .... the polarization from the light scattered by the planet will rotate in angle as the planet orbits the star. So we might detect exoplanets in cases where the traditional methods fail.

The programs discussed here are all on my web page at
http://www.astro.umd.edu/~jph/J_page.html