1.  The Coupled Escape Probability Method in Spherical Symmetry

1.1. Absorption Probability Along a Specific Line-of-Sight

We consider a line with a Doppler profile, so that the (normalized) line profile function for absorption is

\[ \phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad \text{where} \quad x = \frac{\nu - \nu_0}{\Delta \nu_D} \]  

(1)

where \( \Delta \nu_D \) is the Doppler width of the line. Then, with the assumption of complete redistribution, the distribution in frequency of the radiation emitted – by scattering or by thermal processes – is given by the same profile \( \phi(x) \). The optical depth at frequency \( x \) is given by \( \tau \phi(x) \), where \( \tau \) is called the mean optical depth in the line. (Note that the line center optical depth \( \tau(x = 0) = \frac{\tau}{\sqrt{\pi}} \).) Thus the probability that radiation will be emitted at frequency \( x \) and travel optical depth \( \tau \) without absorption is just \( \phi(x) e^{-\tau \phi(x)} \). So we define the function

\[ \eta(\tau) = \int_{-\infty}^{\infty} \phi(x) e^{-\tau \phi(x)} \, dx \]  

(2)

Then, along a particular line-of-sight, the fraction of radiation intercepted between optical depth \( \tau_1 \) and optical depth \( \tau_2 \) will be \( \eta(\tau_1) - \eta(\tau_2) \). This \( \eta(\tau) \) is in some sense analogous to the \( \alpha(\tau) \) of Elitzur and Ramos (2005) (ER05). Note that \( \eta(\tau) \) is a smooth function which can be tabulated and easily interpolated for any \( \tau \). For small values of \( \tau \), a power-series expansion is useful.

1.2. The Line Coupling Matrix for Spherical Shells

Consider a series of spheres of radius \( R_i \) for \( i = 1, 2, ..., (N + 1) \), which bound \( N \) nested spherical shells. Consider a point at radius \( R_i < r_i < R_{i+1} \) in the \( i \)th shell. Let a ray from this point \( r_i \) which makes an angle \( \theta \) with the radial direction (and define \( \mu = \cos \theta \)) ultimately cross the boundaries of shell \( j \) at points \( \tau(\mu, R_j) \) and \( \tau(\mu, R_{j+1}) \). (For some \( \mu \) the line may miss shells \( j < i \). For other \( \mu \)s the line may cut the same shell twice. A line may also cut \( R_{j+1} \) twice, but not \( R_j \). The \( \tau \)’s must be calculated by summing up the segments \( \kappa_k \Delta r(\mu, R_k, R_{k+1}) \) through all the intervening shells. Here, \( \Delta r(\mu, R_k, R_{k+1}) \) represents the distance through shell \( k \) from \( r_i \) along the direction \( \mu \). Then the quantity \( m_{ij}(\mu) = \eta[\tau(\mu, R_j)] - \eta[\tau(\mu, R_{j+1})] \) is the chance that radiation traveling in direction \( \mu \) will be intercepted in shell \( j \). If we then integrate over all angles, we obtain

\[ m_{ij}(r_i) = \frac{1}{2} \int_{-1}^{1} \left[ \eta(\tau(\mu, R_j)) - \eta(\tau(\mu, R_{j+1})) \right] \, d\mu \]  

(3)

the probability that radiation leaving point \( r_i \) in shell \( i \) will be intercepted by shell \( j \). The
value of $m_{ij}$ will vary with the position of $r_i$ within the shell. Thus we must also integrate $r_i$ over the volume of the shell, $dV_i = 4\pi r_i^2 dr_i$, for $R_i < r_i < R_{i+1}$, to obtain

$$M_{ij} = \frac{3}{R_{i+1}^3 - R_i^3} \int_{R_i}^{R_{i+1}} m_{ij}(r_i) r_i^2 dr_i$$

(4)
and we call the array of $M_{ij}$ the coupling matrix. Note that the value $M_{ii}$ is the probability that the radiation is re-absorbed in the same shell from which it was emitted. We have written J code to compute this matrix given a set of shell radii $R_1, ..., R_{N+1}$ and shell opacities $\kappa_1, ..., \kappa_N$.

1.3. The Line Source Function for the Two-Level Atom

Consider the line radiation emitted from a spherical shell $j$ with volume $V_j$. This will be just $4\pi J_j V_j$, where $J$ is the emission coefficient. Now the source function is just $S = J/\kappa$, so the radiation emitted from the shell is $4\pi \kappa_j S_j V_j$. Now the $ji$ element of our coupling matrix $M_{ji}$ is the probability that radiation emitted by shell $j$ will be intercepted by shell $i$, so the radiation emitted by $j$ and scattered in $i$ is $4\pi \kappa_j S_j V_j M_{ji}$.

On the other hand, in terms of the mean intensity $\bar{J}_i$, the radiation scattered in shell $i$ must be $4\pi \bar{J}_i \kappa_i V_i$. If we denote by $\bar{J}_{ij}$ the the mean intensity in shell $i$ which originates in shell $j$, then we can write the radiation emitted in $j$ and scattered in $i$ as $4\pi \bar{J}_{ij} \kappa_i V_i$. Equating this to the expression in the previous paragraph and summing over all emitting shells $j$ we have

$$\kappa_i \bar{J}_i V_i = \sum_{j=1}^{N} \kappa_j V_j M_{ji} S_j$$

(5)
which leads to our expression for the mean intensity in shell $i$:

$$\bar{J}_i = \sum_{j=1}^{N} \left( \frac{\kappa_j}{\kappa_i} \right) \left( \frac{V_j}{V_i} \right) M_{ji} S_j$$

(6)

Now the line source function for the two-level atom is given by

$$S_i = (1 - \epsilon_i) \bar{J}_i + \epsilon_i B_i$$

(7)
so the equation for the source function $S_i$ becomes

$$S_i = (1 - \epsilon_i) \sum_{j=1}^{N} \left( \frac{\kappa_j}{\kappa_i} \right) \left( \frac{V_j}{V_i} \right) M_{ij} S_j = \epsilon_i B_i$$

(8)
or, with $I$ representing the identity matrix, we have the matrix equation

$$
\left[ I_{ij} - (1 - \epsilon_i) \left( \frac{\kappa_j}{\kappa_i} \right) \left( \frac{V_j}{V_i} \right) M_{ij} \right] \times [S_i] = [\epsilon_i B_i]
$$

(9)

1.4. Multi-Level Atoms: The Net Radiative Bracket

The CEP treatment developed by ER05 makes use of the “net radiative bracket” of Athay and Skumanich (ER05, eq. 6):

$$
p(\tau) = 1 - \frac{\bar{J}(\tau)}{S(\tau)}
$$

(10)

From our expression for the mean intensity given above, we thus have

$$
P_i = 1 - \sum_{j=1}^{N} \left( \frac{\kappa_j}{\kappa_i} \right) \left( \frac{V_j}{V_i} \right) M_{ji} \frac{S_j}{S_i}
$$

(11)

This can be inserted into the code we developed for the plane-parallel problems to provide solutions to the corresponding problems in spherical symmetry.

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(1) If $\tau$ is small, a useful expression for $\eta(\tau)$ can be obtained by expanding the exponential in equation (2):

$$
\eta(\tau) = \int_{-\infty}^{\infty} \phi(x) \left\{ 1 - \tau \phi(x) + \frac{\tau^2}{2} \phi^2(x) - \cdots \right\} dx = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} \int_{-\infty}^{\infty} \phi^{n+1}(x) dx
$$

and since

$$
\phi^k = \pi^{-k/2} e^{-kx^2} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-kx^2} dx = \sqrt{\frac{\pi}{k}}
$$

we have

$$
\eta(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^{n/2} n! \sqrt{n+1}} \tau^n
$$

Explicitly, the first few terms are

$$
\eta(\tau) \approx 1 - 0.39894228 \tau + 0.09188815 \tau^2 - 0.01496559 \tau^3 + 0.00188801 \tau^4 - \cdots
$$