Binary black hole merger dynamics and waveforms

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(Dated: February 7, 2006)

We apply recently developed techniques for simulations of moving black holes to study dynamics and radiation generation in the last few orbits and merger of a binary black hole system. Our analysis produces a consistent picture from the gravitational waveforms and dynamical black hole trajectories for a set of simulations with black holes beginning on circular-orbit trajectories at a variety of initial separations. We find profound agreement at the level of 1% among the simulations for the last orbit, merger and ringdown, resulting in a final black hole with spin parameter \(a/m = 0.69\). Consequently, we are confident that this part of our waveform result accurately represents the predictions from Einstein’s General Relativity for the final burst of gravitational radiation resulting from the merger of an astrophysical system of equal-mass non-spinning black holes. We also find good agreement at a level of roughly 10% for the radiation generated in the preceding few orbits.

PACS numbers: 04.25.Dm, 04.30.Db, 04.70.Bw, 95.30.Sf, 97.60.Lf

I. INTRODUCTION

Two black holes in a binary system spiral together due to the emission of gravitational waves. The final merger stage of a binary in which the black holes have comparable masses will produce a spectacular burst of gravitational radiation and is expected to be one of the brightest sources in the gravitational wave sky. Mergers of binaries containing two stellar black holes are important targets for the first-generation LIGO gravitational wave detectors, now operating at design sensitivity in a year-long science data-taking run, as well as other ground-based detectors such as VIRGO and GEO. Knowledge of the waveforms from the final merger phase is important to improve the detectability of these sources by such detectors. Mergers of massive black hole binaries are important sources for the space-based LISA, currently in the formulation phase. Since LISA is expected to observe these massive black hole mergers at relatively high signal-to-noise ratios, comparison of the data with calculated merger waveforms should allow a test of General Relativity in the dynamical, nonlinear regime.

In the early stages of the binary inspiral, the black holes are widely separated and the waveforms can be calculated analytically using perturbative methods. However, waveforms from the final merger, in which the black holes plunge together and form a single, highly-distorted black hole with a common horizon, can only be calculated using full 3-D numerical relativity simulations of the full Einstein equations. This proved to be a very difficult undertaking and, for roughly the past decade, 3-D numerical relativity simulations have been beset by pernicious numerical instabilities that prevented the simulation codes from running long enough to evolve any significant fraction of a binary orbit.

Recently, however, dramatic progress has been made in evolving the merger of equal mass binary black holes. Using excision to remove the singular regions within the horizons and a corotating coordinate system to keep the black holes fixed in the numerical grid, a binary has been evolved through a little more than an orbit, and through merger, though without being able to extract gravitational waves. Simulations of excised black holes allowed to move through the grid on one or more orbits and then through merger have been carried out, with the extraction of gravitational radiation. In addition, new techniques allowing the black holes to move without the need for excision have been developed independently by the authors of this paper and another research group; these have been applied to study the final plunge, and merger of a binary, with the calculation of gravitational waves. Recently, these techniques have been applied to evolve a binary with nonequal masses and the last orbit and merger of an equal mass binary. It is especially noteworthy that this progress is occurring on a broad front, by several independent groups using different techniques.

Several major open questions in the area of binary black hole mergers center on the dependence of the resulting gravitational waveforms on the initial data. In order to use numerical relativity simulations to compare with data from gravitational wave detectors, we need to model astrophysically realistic binary black hole configurations. For non-spinning equal mass black holes, how strongly do the gravitational waveforms for the merger depend on the initial data? What are the effects of spin and non-equal masses on the resulting waveforms? The answers to such questions can only be approached using an evolutionary analysis, in which different initial data sets are evolved and the resulting waveforms are compared.

In this paper, we take a step towards answering these questions by evolving several initial data sets for non-spinning equal mass black holes through the final few orbits, plunge, merger and ringdown. To accomplish this, we apply our new methods that allow puncture black holes to move freely across a grid. Using adaptive mesh refinement, we can resolve the dynamical regions near
the black holes (having length scales \( \sim M \), where \( M \) is the total mass of the system and we use \( c = G = 1 \)) and the outer regions where the gravitational waves are extracted (having length scales \( \sim (10 - 100)M \)). Putting the outer boundary typically at \( t \approx 768M \), causally disconnected from the dynamical regions and wavezone throughout most of the simulation, we can evolve stably for \( t > 800M \). The initial data sets are chosen to be puncture data [11].

Our study focuses on simulations beginning from a set of four inspiralling black hole configurations modelled as described in section [11] and using techniques discussed in section [11]. In section [11] we calibrate the performance of our numerical techniques with a resolution study of simulations from the configuration with the shortest initial separation. Our main results are presented in section [11] where we comparatively study the radiation waveforms and black hole trajectories. We find a close relationship between the trajectory information and the waveforms, providing a consistent picture of the black hole dynamics. The radiation from the final orbit, merger and ring-down agrees to high precision among runs from our range of initial configurations, with the runs from the farthest initial separation producing a promising waveform approximately through the last three orbits.

### II. INITIAL DATA

We start by setting up initial data for equal mass binary black holes represented as “punctures” [11]. The metric on the initial spacelike slice is written in the form

\[
g_{i j} = \psi^4 \delta_{i j},
\]

where \( i, j = 1, 2, 3 \), and the conformal factor \( \psi = \psi_{BL} + u \). The static, singular part of the conformal factor takes the form

\[
\psi_{BL} = 1 + \sum_{n=1}^{\infty} m_n / 2 |\vec{r} - \vec{r}_n|,
\]

where the \( n \)th black hole has mass \( m_n \) and is located at coordinate position \( \vec{r}_n \). The nonsingular function \( u \) is calculated by solving the Hamiltonian constraint equation using AMR-MG [12].

The punctures are initially placed on the \( y \)-axis in the equatorial \( (z = 0) \) plane. We need to specify the individual puncture mass \( m \), coordinate position \( Y \), and momentum \( P \) so that the black holes are on approximately circular orbits and have no individual spins (they are irrotational). To accomplish this, we adapt the quasicircular initial data from the QC sequence given in Ref. [12]. These data are based in turn on the results of Cook [14]; see also [15], who uses an effective potential method that minimizes the binary system energy while holding the orbital angular momentum fixed, to produce an approximately circular orbit.

The parameters and physical quantities for the simulations we ran are shown in Table I. Here, \( Y \) is the initial coordinate position of each puncture along the \( y \)-axis, \( P \) is the linear momentum of an individual puncture, and \( m \) is the mass of the puncture. \( M_0 \) is the initial total ADM mass of the binary, \( J_0 \) is the initial total angular momentum. \( L \) is the initial proper separation of the black holes, approximated by

\[
L = \int H_i \sqrt{g_{yy}} \, dy,
\]

where each limit \( H_i \) represents a point on the Schwarzschild horizon associated with mass \( m_i \) at position \( r_i \).

<table>
<thead>
<tr>
<th>Run</th>
<th>( \pm Y )</th>
<th>( \pm P )</th>
<th>( m )</th>
<th>( M_0 )</th>
<th>( J_0 )</th>
<th>( L/M_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>3.257</td>
<td>0.133</td>
<td>0.483</td>
<td>0.996</td>
<td>0.868</td>
<td>9.9</td>
</tr>
<tr>
<td>R2</td>
<td>3.776</td>
<td>0.119</td>
<td>0.488</td>
<td>1.001</td>
<td>0.899</td>
<td>11.1</td>
</tr>
<tr>
<td>R3</td>
<td>4.251</td>
<td>0.109</td>
<td>0.49</td>
<td>1.002</td>
<td>0.928</td>
<td>12.1</td>
</tr>
<tr>
<td>R4</td>
<td>4.77</td>
<td>0.101</td>
<td>0.492</td>
<td>1.003</td>
<td>0.959</td>
<td>13.2</td>
</tr>
</tbody>
</table>

**TABLE I**: Initial data parameters and physical quantities for the runs considered in this paper.

We note that our runs R1, R2, R3, and R4 correspond closely to the initial data for models QC-6, QC-7, QC-8, and QC-9, respectively, along the QC sequence. As shown in Ref. [15], at these relatively wide initial separations, the parameters for these initial data sets are close to those derived using post-Newtonian (PN) techniques; see in particular Figs. 26 - 28 in Ref. [12].

### III. SIMULATION TECHNIQUES

We evolve the initial data with the **Hahndol code** [16], which uses a conformal (BSSN) formulation of Einstein’s evolution equations on a cell-centered numerical grid. The basic equations are the same as given in Ref. [16], with the exception that the evolution equation for the BSSN variable \( \tilde{\Gamma} \) has been modified for stability as suggested in [17]. In addition, to reduce high frequency noise associated with refinement interfaces, we add some dissipation of the form given in [18].

Mesh refinement and parallelization are implemented in our code with the **PARAMESH** package [19, 20]. We use 4th order centered differencing for the spatial derivatives except for the advection of the shift, which is performed with 4th-order upwinded differencing. The refinement boundary interfaces are buffered with 4th-order-interpolated guard cells which, at worst, may introduce 2nd-order errors into second derivatives. With our current mesh refinement implementation, our accuracy is limited by spatial finite differencing error from refinement interfaces. Since we do not gain much by using higher order time integration, we use 2nd-order time stepping via a three-step iterative Crank-Nicholson scheme. Even though, overall, we expect 2nd-order convergence, we have found considerable advantage in using 4th-order spatial differencing over 2nd-order spatial differencing, as measured by the accuracy and manifest convergence of the Hamiltonian constraint and other quantities.

Traditionally, puncture black holes have been evolved by keeping them fixed in the grid [3, 4]. This is accomplished by factoring out the singular part \( \psi_{BL} \) and handling it analytically, while evolving only the regular parts of the metric. As explained in detail in [3], we employ instead newly developed techniques that allow the puncture...
black holes to move freely across the numerical grid. The
singular part is not factored out; instead the entire con-
formal factor is evolved. Initially, the binary is set up so
that the centers of the punctures are not located at grid
points (as in the traditional implementation). Taking
numerical derivatives of \( \psi_{\text{BH}} \) causes an effective regular-
ization of the puncture singularity through the inherent
smoothing of finite differences.

Our new approach allows the punctures to move freely
by implementing a modified version of the Gamma-
freezing shift vector. The Gamma-freezing condition

generally improves numerical stability by evolving the co-
dinates towards quiescence, in accord with the physical
dynamics. Our modification to this gauge is tailored for
moving punctures, as mentioned in \[7\]. Specifically we use

\[
\partial_t \beta^i = \frac{3}{4} \alpha B^i
\]

and

\[
\partial_t B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i
\]

for the shift \( \beta^i \) and

\[
\partial_t \alpha = -2\alpha K + \beta^j \partial_j \alpha
\]

for the lapse \( \alpha \). The lapse equation \( \partial_t \alpha \) is the well-known
“1+log” slicing condition, modified with an advection term as in \[8\]. We use the initial gauge conditions \( \beta = 0, \quad B = 0, \quad \alpha = \psi^{-2}, \) similar to those recommended in \[8\]. We place the punctures in the \( z = 0 \) plane and impose equatorial symmetry throughout.

We use adaptive mesh refinement to produce a numer-
cal grid having appropriate resolution in the strong-field
regions near the black holes and in the wave zone.
Initially, we set up the black hole binary in a nu-
merical domain with a box-in-box refinement structure
having an innermost refinement of \( h_f \). We generally extract gravitational waves on a
this box and the next (fixed) one at \( \pm 48M \) in all runs. We generally extract gravitational waves on a
sphere of radius \( r_{\text{ex}} = 30M \). In the early stages of the run, this sphere intersects the next innermost box with
resolution \( h_w/2 \); at later times, it is completely located
within the region having resolution \( h_w \).

We extract gravitational waves from our simulations
using the Weyl tensor component \( \Psi_4 \). Our wave extraction techniques are based on Misner's method \[21\] and are 2nd-order accurate. They are robust and accurate even when the extraction radii cross mesh refinement bound-
aries \[22\]. In particular, the waveforms computed at vari-
ous extraction radii \( r_{\text{ex}} \) are preserved up to the leading
order \( 1/r \) scaling and show no ill effects from passing
through one or more refinement boundaries.

IV. CALIBRATION OF SIMULATIONS

We have performed detailed studies of the errors and convergencen behavior of simulations run on the
\texttt{Hahndol} code using fixed mesh refinement in previous work \[7, 16, 22, 23\]. The simulations presented here differ
from this earlier work in two important ways: they
are carried out using an adaptive mesh structure that
changes as the binaries evolve, and they are run for sig-
ificantly longer durations. In this section, we discuss
the calibration tests we have carried out to verify that
the code produces robust, reliable results in these more
challenging regimes.

We performed these calibration tests on run R1, which
we ran at three different resolutions. The initial data
parameters for R1 are given in Table \[I\] and the simulation
parameters for these three cases are shown in Table \[II\].
We use a base resolution of \( \rho = 3M/32 = 0.09375M \) from
which we reach the three resolutions used in the runs.
Note that the medium resolution case, \( h_f = \rho/3 \), has the

<table>
<thead>
<tr>
<th>R1 cases</th>
<th>low</th>
<th>medium</th>
<th>high</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_f )</td>
<td>( \rho/2 )</td>
<td>( \rho/3 )</td>
<td>( \rho/4 )</td>
</tr>
<tr>
<td>outer boundary</td>
<td>( \pm 768M )</td>
<td>( \pm 192M )</td>
<td>( \pm 768M )</td>
</tr>
<tr>
<td>( T_{\text{sim}} )</td>
<td>186M</td>
<td>332M</td>
<td>291M</td>
</tr>
</tbody>
</table>

\[1\] For convenience, we use \( M = 1 \) to set the scale for the compu-
tational grid and time. Note that the actual initial total ADM
mass for each case is \( M_0 \approx \rho \), as given in Table \[II\].
outer boundary located relatively close, at $\pm 192M$; this was one of the earliest runs we did. Since the simulation runs for a total duration $T_{\text{sim}} = 332M$, small errors from this outer boundary do have time to propagate into the physically interesting regions of the grid. While these effects are generally small and have no significant impact on the dynamics or wave extraction, we did use a more distant outer boundary at $\pm 768M$ in all other runs to eliminate this problem. This adds only a small overhead to the overall cost of the simulation, due to the fixed mesh refinement structure used in the outer regions.

For these three runs, we scaled the criterion for refinement and derefinement to provide as close as possible the same grid structure at the same physical time in the simulations. Since it is generally not possible to match the grid structures exactly, pointwise convergence tests are of questionable value. Instead, we calculated the $L^1$ norm of the Hamiltonian constraint $C_H$ over the grid as a function of time. This norm was taken over all levels inside the box at $\pm 48M$, which includes the wave extraction zone. Figure 1 shows the $L^1$ norms for these three runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50 runs; note that the initial growth of the constraint violation is brought to a halt after approximately 50.

The gravitational waves are extracted on a sphere of radius $r_{\text{ex}} = 30M$. The top panel of Fig. 2 shows the $l = 2$, $m = 2$ component of $\Psi_4$, for the three different resolution runs, where $r$ is the estimated areal radius of the extraction sphere. The waves start out in phase; note in particular the agreement in the initial pulse around $t \sim 30M$. However, as the runs proceed, timing differences slowly accumulate over the relatively long orbital time scales. By $t \sim 150M$, significant differences have accumulated in the lowest resolution case (solid line) compared to the other two runs; we attribute this to the larger inherent numerical diffusion in this lowest resolution run.

A naive convergence test, in which we take differences directly between the waveforms at the same simulation times, is shown in the bottom panel of Fig. 2. These curves are scaled so that, for $2^{\text{nd}}$—order convergence they would lie on top of each other. As Fig. 1 shows, we get $2^{\text{nd}}$—order convergence (or slightly better) for the entire course of the run.

The gravitational waves are shown as a function of time for the three different resolution runs of R1 given in Table II. The high resolution case is shown with a dashed line. The medium (bold dashes) and low (solid) cases are scaled so that, for $2^{\text{nd}}$—order convergence all three curves would lie on top of each other. The $L^1$ norm is taken only over all levels of the grid inside $48M$, including the wave extraction region. This figure indicates satisfactory convergence of the Hamiltonian constraint error in our simulations.

A more meaningful convergence test is shown in Fig. 3. Here, we have shifted the waveforms in time to adjust for the phase differences using the techniques described below. The time axis has been relabeled so that the peak

TABLE II: Parameters for the low, medium, and high resolution runs of model R1, where $\rho = 3M/32 = 0.09375M$. $T_{\text{sim}}$ is the total duration of the simulation.
of the radiation occurs at \( t = 0 \). The top panel shows the \( l = 2, m = 2 \) mode of \( r\Psi_4 \) for the low (solid), medium (bold dashes), and high (dashes) resolution cases; note that the physical parts of the waveforms \((t \gtrsim -125M)\) agree beautifully. We then carried out a convergence test using these shifted curves; the results are shown in the bottom panel. Again, the curves are scaled so to lie on top of each other for 2nd-order convergence. Here we see nearly perfect convergence for the physical parts of the waveforms; note in particular the much smaller vertical scale in the lower panel of Fig. 3.

![Time-shifted gravitational waveforms and a physical convergence test.](image)

**FIG. 3:** Time-shifted gravitational waveforms and a physical convergence test. The labels in the top level are as in Fig. 1. In the top panel, the gravitational waveforms have been shifted in time so that the peak amplitude of the radiation occurs at \( t = 0 \). The differences between these time-shifted waveforms are shown in the bottom panel; these curves are scaled so that they would lie on top of each other for 2nd-order convergence.

As shown in Table III, the radiated angular momentum \( J_{\text{rad}} \) varies by a few percent between \( r_{\text{ex}} = 30M \) and \( r_{\text{ex}} = 50M \) at high resolution. This observation seems to agree with the notion that the angular momentum depends more strongly on the longer wavelength parts of the waves \(13\), which should be extracted at greater distances. For all the runs reported in this paper, we use an extraction radius of \( r_{\text{ex}} = 50M \) to calculate \( J_{\text{rad}} \).

### Table III: Values of the energy \( E_{\text{rad}} \) (in units of \( M_0 \)) and angular momentum \( J_{\text{rad}} \) (in units of \( M_0^2 \)) carried away by gravitational radiation for the R1 runs calculated for different extraction radii and different resolutions.

<table>
<thead>
<tr>
<th>( h_f )</th>
<th>( r_{\text{ex}} = 20M )</th>
<th>( r_{\text{ex}} = 30M )</th>
<th>( r_{\text{ex}} = 40M )</th>
<th>( r_{\text{ex}} = 50M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho/2 )</td>
<td>( E_{\text{rad}} )</td>
<td>0.0347</td>
<td>0.0343</td>
<td>0.0336</td>
</tr>
<tr>
<td></td>
<td>( J_{\text{rad}} )</td>
<td>0.217</td>
<td>0.218</td>
<td>0.218</td>
</tr>
<tr>
<td>( \rho/3 )</td>
<td>( E_{\text{rad}} )</td>
<td>0.0343</td>
<td>0.0345</td>
<td>0.0345</td>
</tr>
<tr>
<td></td>
<td>( J_{\text{rad}} )</td>
<td>0.215</td>
<td>0.223</td>
<td>0.226</td>
</tr>
<tr>
<td>( \rho/4 )</td>
<td>( E_{\text{rad}} )</td>
<td>0.0342</td>
<td>0.0344</td>
<td>0.0345</td>
</tr>
<tr>
<td></td>
<td>( J_{\text{rad}} )</td>
<td>0.216</td>
<td>0.224</td>
<td>0.227</td>
</tr>
</tbody>
</table>

As we discussed in 13, the gravitational waveforms can be used to calculate the total energy \( E_{\text{rad}} \) and total angular momentum \( J_{\text{rad}} \) carried away by the radiation. We calculate \( dE/dt \) and \( dJ/dt \) from time integrals of all \( l = 2 \) and \( l = 3 \) waveform components using Eqs. (5.1) and (5.2) in 13. Integrating \( dE/dt \) gives the total energy loss due to the radiation, \( E_{\text{rad}} \). We find that the influence of higher modes of the waves contribute < 1% to the energy and even less to the angular momentum.

### V. RESULTS

In this section we comparatively analyze the results of simulations based on our R1-R4 black hole configurations. Recall that the initial data for each of these simulations starts the black holes at different separations, with proper separations varying from 9.9\( M_0 \) to 13.2\( M_0 \), and provided with sufficient angular momentum that the runs are estimated to be on initially circular orbits.

We can think of each of these simulations as an approximate representation of the late-time portion of an ideal spacetime which begins with arbitrarily well-separated black holes on an inspiraling trajectory which asymptotically approaches circular orbits. As such representations, there are several limitations which the initial data models may have. In particular, the initial parameters will only approximate the ideal trajectory since the angular momentum as a function of radius may differ from the value required for an idealized circular orbits. Likewise, at finite radius, the ideal spiral trajectory can only be approximated by our initially circular configurations.

Furthermore the manner by which we have mapped from these trajectory specifications to actual initial data values necessarily requires making some suppositions. For instance, our puncture data prescription, like almost all field prescriptions, contains no representation of prior radiation generated before the time at which the initial data are posed. Though it is generally expected that the significance of such limitations on the final merger simulations should be reduced if the black holes begin...
sufficiently far apart, there is no clear way to assess just how significant such effects will be on the results, including the gravitational waveforms, before carrying out the evolutions.

Our simulations, begun with varying initial separations, should be affected by any initial modeling error in varying amounts, but should agree to the degree that they represent the ideal astrophysical spacetime. A key objective in our analysis is to identify universal characteristics among the different runs which, we reason, are then likely to correctly represent those aspects of the astrophysical equal-mass binary black hole merger spacetime.

A. Overview of Simulations

Our comparative analysis covers four simulations labeled R1 to R4 in Table IV. We evolved them all using the medium resolution of \( h_f = \rho/3 \) except for R1, where we used the higher \( h_f = \rho/4 \) resolution. In all runs we used an initial grid setup and adaptive mesh refinement as described in Sec. III. We evolved all the runs to well after the wave signal had passed the extraction region. The actual amount of time is noted as \( T_{\text{sim}} \) in Table IV.

For the time-slicing condition used in our simulations, the region where the lapse satisfies the condition \( \alpha = 0.3 \) corresponds roughly with the apparent horizon location. We thus used the moment when the two \( \alpha = 0.3 \) regions around the black holes merge to specify a merger time \( T_{\text{merge}} \). The number of orbits for each run, \( N_{\text{orbits}} \), was estimated from the trajectories shown in Fig. 4 and is taken up to the point at which the merger occurs.

<table>
<thead>
<tr>
<th></th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L/M_0 )</td>
<td>9.92</td>
<td>11.1</td>
<td>12.1</td>
<td>13.2</td>
</tr>
<tr>
<td>( h_f )</td>
<td>( \rho/4 )</td>
<td>( \rho/3 )</td>
<td>( \rho/3 )</td>
<td>( \rho/3 )</td>
</tr>
<tr>
<td>( T_{\text{sim}} )</td>
<td>421M</td>
<td>531M</td>
<td>530M</td>
<td>850M</td>
</tr>
<tr>
<td>( T_{\text{merge}} )</td>
<td>160M</td>
<td>234M</td>
<td>396M</td>
<td>513M</td>
</tr>
<tr>
<td>( N_{\text{orbits}} )</td>
<td>1.8</td>
<td>2.5</td>
<td>3.6</td>
<td>4.2</td>
</tr>
</tbody>
</table>

TABLE IV: Simulation parameters and results. \( T_{\text{merge}} \) is the time at which the merger occurs, starting from the initial time in each run.

A graphical overview of our four simulations is presented in Fig. 4 showing the paths traced by the black hole punctures on the computational domain. These were obtained by numerically integrat

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\[
\begin{array}{cccc}
L/M_0 & R1 & R2 & R3 & R4 \\
9.92 & 11.1 & 12.1 & 13.2 \\
\rho/4 & \rho/3 & \rho/3 & \rho/3 \\
421M & 531M & 530M & 850M \\
160M & 234M & 396M & 513M \\
1.8 & 2.5 & 3.6 & 4.2 \\
\end{array}
\]

TABLE IV: Simulation parameters and results. \( T_{\text{merge}} \) is the time at which the merger occurs, starting from the initial time in each run.

A graphical overview of our four simulations is presented in Fig. 4 showing the paths traced by the black hole punctures on the computational domain. These were obtained by numerically integrating the equation of motion \( \vec{x}_{\text{punc}} = -\beta(\vec{x}_{\text{punc}}) \), which analytically gives the exact trajectory of each puncture. The \( \beta(\vec{x}_{\text{punc}}) \) was interpolated between grid points as required.

For clarity, Fig. 4 shows only the track of one of the two black holes from each simulation. We have oriented each trajectory according to a physical reference discussed in Sec. V.B so that they superpose at the radiation peak, which occurs very near the end of the puncture trajectory. R4 has the widest initial separation and completes the largest number of orbits. Each of the other cases, after an initial transient period of approximately one orbit, nearly locks on to the R4 trajectory. For the final orbit, all four trajectories are very nearly superposed. In Sec. V.C we study the quantitative features of these trajectories in more detail.

FIG. 4: Paths of black holes starting from different initial separations. The paths are very similar for approximately the last orbit, indicating that the black holes follow the same tracks. The point of merger (estimated by a single connected isosurface of \( \alpha = 0.3 \)) is indicated by an asterisk in the plot.

B. Gravitational Radiation

The first step in comparing our runs is to calculate the energy and angular momentum carried away by the gravitational radiation generated by the binary system in our simulations. Following the discussion in Sec. IV, we
measure the radiation energy extracted at \( r_{\text{ex}} = 30M \), estimating that this will be accurate within \( \sim 1\% \). We subtract the radiation energy \( E_{\text{rad}} \) from the initial mass \( M_0 \), given in Table II, to determine the final black hole mass in each simulation, \( M_f = M_0 - E_{\text{rad}} \). This provides a physical scaling which we use to compare the R1-R4 simulations. Similarly, we measure the angular momentum content of the radiation \( J_{\text{rad}} \) as extracted at \( r_{\text{ex}} = 50M \), estimating the accuracy to be within a few percent. Subtracting this from the initial ADM angular momentum \( J_0 \), we calculate the spin parameter of the final black hole \( a/M = (J_0 - J_{\text{rad}})/M_f^2 \). The results are summarized in Table V.

<table>
<thead>
<tr>
<th>( E_{\text{rad}}/M )</th>
<th>( J_{\text{rad}}/M^2 )</th>
<th>( a/M )</th>
<th>( M_{\text{QN}}/M )</th>
<th>( a_{\text{QN}}/M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1 0.0356</td>
<td>0.246</td>
<td>0.694</td>
<td>1.005</td>
<td>0.721</td>
</tr>
<tr>
<td>R2 0.0369</td>
<td>0.272</td>
<td>0.691</td>
<td>1.002</td>
<td>0.686</td>
</tr>
<tr>
<td>R3 0.0381</td>
<td>0.306</td>
<td>0.689</td>
<td>1.004</td>
<td>0.694</td>
</tr>
<tr>
<td>R4 0.0387</td>
<td>0.325</td>
<td>0.702</td>
<td>1.004</td>
<td>0.693</td>
</tr>
</tbody>
</table>

TABLE V: Energy and angular momenta for the radiation and final black hole. \( E_{\text{rad}} \) and \( J_{\text{rad}} \) are measured at \( r_{\text{ex}} = 30M \), and \( r_{\text{ex}} = 50M \), respectively. \( M_{\text{QN}} \) and \( a_{\text{QN}} \) are calculated independently from the quasi-normal fits of the ringdown waveforms, and agree well with the values deduced from the radiative losses.

We note that energy and angular momentum content of the radiation is almost entirely contained in the \( l = 2 \) \( m = \pm 2 \) spin-2-weighted spherical harmonic components, with other components entering at the 1% level. In the remainder of our waveform analysis we concentrate exclusively on the leading component,

\[
r_{\Psi_4}(\theta, \phi) \sim r\Psi_{4(22)}(-2Y_{22}(\theta, \phi) + -2Y_{2-2}(\theta, \phi))
\]

where, for simplicity, we have suppressed retarded time dependence. Hereafter we suppress the multipole labels and refer to the leading component simply as \( r\Psi_4 \). The two polarization components of the radiation are represented in the real and imaginary parts of \( r\Psi_4 \). We follow Refs. 13, 24, representing our waveforms by \( r\Psi_4 = A \exp(-i\varphi_{\text{pol}}) \), where the amplitude \( A \) and polarization phase \( \varphi_{\text{pol}} \), like \( r\Psi_4 \), are functions of time. Of course, any complex time-series could be represented in this form, but it is particularly valuable if the radiation exhibits a circular polarization pattern, meaning that \( A \) and \( \varphi_{\text{pol}} \) vary slowly compared to the wave frequency timescale. Such radiation will be circularly polarized to an observer on the system’s rotational axis, varying to linearly polarization for an observer on the equatorial plane. We find that the radiation produced in our simulations shows strong circular polarization, which we will utilize in comparing the radiation from our four simulations.

As we are interested in the degree to which our various runs may be taken to be different models of the same physical merger spacetime, we must define a physical basis for comparing them. As physically expected, runs beginning with more separated black holes take a longer time to reach the point of merger. For each run, a physical reference time is recognizable by the point at which the radiation reaches its peak amplitude; we define \( t = 0 \) at this point for our comparisons.

Fig. 6 shows the wave amplitudes \( A(t) \) from all four runs. Through the strong radiation peak after \( t = -50M \) all four wave amplitudes show striking universality in the compared results, with agreement among all runs to about 1%. This period of strong agreement covers roughly the last orbit, plunge and ringdown of the merger. Agreement within about 10% is maintained among the R2-R4 simulations for most of each run with slightly more difference in the R1 run. The smooth shape of the peak, lacking any sign of the several wave cycles spanning the peak (see Fig. 5) is an indication of the circular polarization pattern discuss above. The only

FIG. 5: Waveforms from runs R1 - R4. The first 500M are shown in an inset to emphasize the initial data transient part.

FIG. 6: Amplitudes (absolute value of (complex) \( \Psi_4 \)) of the waves. The curves have been shifted such that the maxima are all at time 0. The inset zooms into the peak showing the strong agreement from \( t = -50M \) on.
FIG. 7: The phase angle vs. time. The phase is made to agree at time 0M. But after that the phases agree very well showing that the ringdown is the same for all the runs. And it agrees before that time showing that the black holes follow the same dynamics independent of their initial separation.

FIG. 8: Waveform frequency as a function of time. The agreement from $t = -50M$ on is astounding.

The final (constant) frequency is the frequency of the ringdown. Fitting an exponential decay to the amplitudes and using this final frequency we estimate, using the procedure in Ref. [24], the mass and angular momentum of the black hole formed in the merger. These estimates provide a description of the final black hole as determined by its perturbative dynamics. The results are listed in Table. V where they can be compared with the independent estimates obtained by subtracting the radiation losses from the initial values. The good agreement, to one or two percent, provides another measure of energy and angular momentum conservation in each simulation. We also note the strong agreement among our simulations from different separations on the final state of the remnant black hole, with every measure consistent with the same value, $a/M = 0.69(\pm 2\%)$.

C. Trajectory analysis

In this section we consider the particle-like dynamics of our simulations defined by the coordinate trajectories of our black hole punctures. It is important to be careful in interpreting such coordinate-dependent information which may include non-physical gauge features. Nonetheless it is worth noting that the $\Gamma$-freezing gauge condition applied in our simulations, to the extent that it approximates $\Gamma^a = \gamma^{ij} = 0$ is similar to the Dirac gauge applied in some post-Newtonian calculations, and might be expected to provide a sensible coordinate system at least in weak field regions. In any case, since we apply similar coordinate conditions in each run, the coordinate-based puncture trajectories provide another opportunity to identify universal features in our simulations from different initial separations.

Refer again to Fig. 4 which shows the tracks of one of the punctures from each of the runs R1 - R4 as they spiral into the center. As the black holes descend deeper into the strong field, we see that their paths lock on to

clearly non-universal feature in the wave amplitudes is a small and brief burst lasting about 70M at the beginning of each run. We interpret this burst as “spurious radiation” content in the initial data. As generally expected the amplitude of the spurious radiation lessens as the initial separation of the black holes is increased. Since the black holes in our various runs (which all begin on the $y$-axis) undertake differing amounts of orbital motion before merger, we must expect differing orientations for the systems at the point of merger. For meaningful comparisons we must rotate the data from each system to align them with respect to some physical reference. As our reference, we will orient the systems so that the polarization phase $\varphi_{pol}$ of the radiation passes though zero at the moment of peak amplitude $t = 0$. We use this orientation for all figures with wave phase information, and in the trajectory comparison in Fig. 4

The polarization phase $\varphi_{pol}$ is shown in Fig. 7. At time $0M_0$ the phases are set to agree. However, they keep agreeing after that, showing that the ringdown frequency is the same. The plot shows generally good phase agreement among all runs, to within a small fraction of a wave cycle, except for a brief period at the beginning of each run when the radiation is dominated by spurious radiation associated with initial data modeling error, and is not circularly polarized. For circularly polarized radiation it is meaningful to define an instantaneous frequency for the wave, $\omega = \partial \varphi_{pol}/\partial t$, shown in Fig. 8 which allows a more detailed comparison of the simulation waveforms. Clearly can be seen that apart from the noise due to the smallness of the waves initially and at the end, all runs have the same frequency evolution from about 60$M_0$ before the merger. Before that the frequency evolution compares similarly among the simulations with some wavyness which may correspond to some ellipticity in early part of each simulation’s inspirial.
a universal trajectory that takes them into the plunge and subsequent merger. This behavior can also be seen in Fig. 9, which shows the coordinate separation between the punctures, $r = (x^2 + y^2)^{1/2}$ as a function of time. In this case, there is strong agreement among the runs after $t \approx -50M$. In the earlier part of the simulations there are clear differences among the runs, which are suggestive of ellipticity in the initial orbital motion.

One way to estimate the utility of such coordinate information is by comparison with our much more invariant waveform data. We can, for instance, compare the coordinate orbital frequency $\Omega$ with the waveform frequency examined above. With suitable coordinates, in the weak-field limit, we would expect the orbital frequency to be approximately equal to half the gravitational wave frequency. In Fig. 10 we compare the coordinate orbital frequency $\Omega$ with half the gravitational wave frequency from our R4 run. We find, that if we shift the orbital frequency data by about $33M_0$, the two curves match very well. Despite some noise in our wave frequency, most features in the orbital frequency are tracked in the wave frequency as well. Exceptions occur at the very beginning of the simulation, where our coordinates necessarily start with the punctures non-moving (hence $\Omega = 0$) before the brief period through which the coordinate puncture velocity seems to adapt well to the physical dynamics. Likewise, at late times, we see that the orbital puncture frequency continues to grow while the radiation frequency saturates at the quasinormal ringing frequency. This seems to correspond to the expectation that, at late times, the coordinate motion of the punctures decouples from the process of radiation generation, with the punctures continuing to fall into the newly formed black hole while gravitational ring-down radiation is generated in final black hole’s perturbative potential barrier region somewhat outside the horizon.

The $33M_0$ time shift required to realize this agreement can be interpreted as the time required for the gravitational radiation to propagate out to point where it is extracted at $R = 30M_0$. We can utilize this time correspondence to roughly associate phenomena occurring in the strong field region of the simulations with features in the radiation. We have used this time-shift to compare the time at which peak radiation is generated in our simulations with $T_{\text{merge}}$ in Table IV. In our simulations, we find that the merger occurs about $12M$ before the radiation peak is generated.

In Fig. 11 we compare the orbital frequencies of the puncture trajectories among our R1-R4 runs. As was the case with the wave frequencies we see excellent agreement among the runs after $t = -50M$. Here we have focused on the earlier region where the results are not quite as universal. In each run we notice that after a period of initial angular acceleration the orbital frequencies agree to

\[ \frac{1}{10^1} \]
within a few percent. This initial transient period is at least partially a coordinate effect (since $\Omega$ must begin at 0), but the strong similarity with wave-frequencies suggests that there may be a physical basis for the discrepancies as well. Particularly for R3 and R4, the shape of the curves suggests a slow oscillation, perhaps some ellipticity in the motion. For an external comparison, we have included the post-Newtonian frequency evolution 1.5PN and 2PN order, as provided in Ref. [26], positioned in time so that all curves agree near $t = -200 M$. The correspondence with these PN curves is certainly striking, though we caution that the 2.5PN curve does not agree so well, with its peak frequency topping out too low for a useful comparison here. In subsequent work we intend to explore comparisons with post-Newtonian calculations in more detail.

VI. DISCUSSION

We have presented a set of numerical simulations representing the last few orbits and merger of an equal-mass non-spinning binary black hole system. Though the initial data differ, with the black holes starting out at separations ranging from $9 M$ to $13 M$, the calculated waveforms are dominated by universal characteristics. Over the period beginning from about $50 M$ before the gravitational wave peak, and covering approximately the last orbit, all our simulations show profound agreement, with differences among the waveforms at no more than about 1%. Consequently, we are confident that this part of our waveforms accurately represents the predictions from Einstein’s General Relativity for the final burst of gravitational radiation resulting from the merger of an astrophysical system of equal-mass non-spinning black holes.

We also see good agreement among our simulations in the gravitational radiation generated in the several preceding orbits which we have simulated. Excepting an initial transient period about $100 M$ for each run, the earlier portion of the waveforms show good agreement, within approximately 10% in phasing and amplitude. For our longest simulation (R4) this suggests that we have produced a good approximation of the astrophysical waveform prediction covering more than $400 M$ before the radiation peak. We are further encouraged by good agreement over much of this period among the frequency evolution of our simulated waveforms (especially R4), with the second-order (2PN) post-Newtonian predictions. We are currently exploring the correspondence of our waveforms with post-Newtonian calculations, to be developed in more detail in a future publication.

That the late-time part of the simulations (the final orbit and thereafter) shows stronger universality than the earlier part of the simulations supports idea that the late dynamics of the system is dominated by the strong interaction of the holes, and radiative losses, which have the effect of reducing dependence on prior conditions. For the remnant black hole formed in the merger, our simulations consistently predict a spin within about 1% of $a/m = 0.69$.

Our simulations have employed newly developed numerical relativity techniques for evolving black holes, which allows the black holes to propagate accurately across the numerical domain. This approach does not require excision of the black hole interior from the computational domain. We have calibrated our approach on one of our black hole configurations, R1, demonstrating 2nd-order convergence, and waveform accuracy at the 1% level in the last orbit. For all our simulations we have found good energy and angular momentum conservation as measured by comparing the mass and spin of the final black hole, as measured by its quasinormal ringing with the expected remainder after radiative losses. In future work we expect to continue to refine our techniques for accuracy and efficiency over long-lasting simulations.

We have also studied the trajectories traced out by the motion of the black holes in our numerical coordinate system. These trajectories suggest a coherent picture of the system’s evolution which is qualitatively, and in some ways, quantitatively consistent with invariant information measured in the radiation waveforms. This correspondence provides some foundation for giving a tentative physical interpretation to some coordinate-based information in our simulations, such as the number and rate of inspiral orbits and recommends further research toward a deeper understanding the properties of the coordinate systems which we have applied in these simulations.

Our comparative analysis provides some insight into the quality of the initial data models which we have applied. We have seen an indication of a small amount (with negligible energy content) of spurious radiation arising from initial modeling error. As generally expected the scale of this spurious radiation decreases for increasingly well-separated initial data. For sufficiently long-lasting runs this spurious radiation is well-segregated in time, limiting its direct significance in interpreting the merger radiation. Of perhaps greater concern are indications which suggest ellipticity in the inspiral trajectories and waveforms. Two aspects of our initial data model may be contributing to this effect. First, the will be some level of mismatch between the initial separation of the black holes and the specified initial value of angular momentum. We comment, in this regard, that other simulations we have looked at in preparation for the work presented here hint that the early transient part of each simulation, as well as measures such as the merger times, may depend sensitively on small (1%) changes in the initial angular momentum. Secondly, as is customary, the data we have applied have the black holes set on initially circular trajectories, with vanishing radial momentum components. This may also be expected to lead to ellipticity as has been noted in binary neutron star simulations. We expect to explore these and similar concerns in future simulations.

Taken together with other recent progress in numeri-
cal relativity, these results herald a new age of numerical simulations applied to further characterize and understand strong-field binary black hole interactions and merger radiation. Future applications will begin to explore the physical parameter space of these systems, to study, particularly, the effect on the radiation waveforms of individual black hole spin, and variations in the black hole mass ratio.

Acknowledgments

We thank David Brown for providing AMRMG, and Cole Miller for many insightful discussions. This work was supported in part by NASA grants ATP02-0043-0056 and 05-BEFS-05-0044. The simulations were carried out using Project Columbia at NASA Ames Research Center and at the NASA Center for Computational Sciences at Goddard Space Flight Center. M.K and J.v.M. were supported by the Research Associateship Programs Office of the National Research Council and the NASA Postdoctoral Program at the Oakridge Associated Universities.