N-body problem, continued

Consider now a system of particles (e.g., stars or dark matter particles) that has reached virial equilibrium. What happens next? In the short term, nothing, since that's what equilibrium means. Let us think, however, about the motion of an individual particle. First, suppose that the cluster has mass that is distributed perfectly smoothly, in a sphere. Place a particle at the edge of this sphere, with no speed, and let it go. Remembering that there are no collisions, **Ask class:** what will happen to the particle? It will fall through the center and come out the other side, slow down, stop at a radius equal to what it had before, and return. If the distribution really is perfectly smooth, this will continue indefinitely. On the other hand, if the mass is concentrated into one big star, then our particle will get deflected significantly by passing close to the star.

What this means is that any change in the motion of a particle is due to its gravitational interactions with *individual* particles. Each interaction deflects our chosen particle a bit, until after some large number of encounters the velocity of our particle has changed by a large amount (e.g., of order itself). This process of scrambling velocities by individual encounters is called two-body relaxation, because it is interactions of two objects that are most important (a three-body or higher interaction between single point masses would require an improbable coincidence to have all those objects nearby at once). But how long does such two-body relaxation take?

To answer this, consider an individual two-body encounter. Let a star of mass m pass another star of mass M, with an initial relative speed v. The impact parameter is the closest that the two stars would come to each other if they moved in straight-line paths; call the impact parameter b. Suppose that this is a weak encounter, so that the deflection $\delta\theta$ of the trajectory is small. Let the initial motion of m be in the $\hat{x}$ direction. Then the acceleration as a function of time t (where $t = 0$ at closest approach) is

$$\ddot{\mathbf{r}} = -\frac{GMb}{(v^2t^2 + b^2)^{3/2}}\hat{y} - \frac{vtGM}{(v^2t^2 + b^2)^{3/2}}\hat{x} \; . \tag{1}$$

The change in velocity is this acceleration integrated over all time. Therefore,

$$\Delta v_x = \int_{-\infty}^{\infty} \ddot{x}dt = -\int_{-\infty}^{\infty} \frac{vtGMdt}{(v^2t^2 + b^2)^{3/2}} = 0 \tag{2}$$

where the integral vanishes because it is antisymmetric in t. Put another way, for a small deflection $\delta\theta$, the new x velocity will be roughly $\cos(\delta\theta)$ times the old x velocity. For $\delta\theta \ll 1$, $\cos(\delta\theta) \approx 1 - \mathcal{O}[(\delta\theta)^2] \approx 1$. Therefore, there is only a tiny change in the x component of the velocity. The change in the y component is

$$\Delta v_y = -\int_{-\infty}^{\infty} \frac{GMbdt}{(v^2t^2 + b^2)^{3/2}} = -\frac{2GM}{vb} \; . \tag{3}$$

Therefore, the deflection angle is $\delta\theta = |\Delta v_y|/v = 2GM/(v^2 b)$.

How small is this angle? It depends on the impact parameter. If the impact parameter is comparable to the separation between stars, then $b \sim n^{-1/3}$, so $\delta\theta \sim 2GMn^{1/3}/v^2$. If the system is virialized, then $v^2 \sim GNM/r$, where N is the number of stars of mass M. Also, the number density is $n \sim N/r^3$ for a cluster radius r. Then,

$$\delta\theta \sim \frac{2GM}{GMN/r}\frac{N^{1/3}}{r} \sim N^{-2/3} \ . \tag{4}$$

For large N, $\delta\theta$ is tiny; for example, a globular cluster might have $N \sim 10^6$ stars, so $\delta\theta \sim 10^{-4}$, or only a few tens of arcseconds. You can similarly convince yourself that the fractional change in velocity is $|\Delta v_y|/v \sim N^{-2/3}$.

Therefore, an individual encounter doesn't do much to the velocity. In fact, you can convince yourself by symmetry that the average velocity doesn't change if the two particles are of equal mass. That is, for a given impact parameter, any interaction that changes the y component of velocity by $\Delta v_y = -2GM/(vb)$ has a mirror image encounter with $\Delta v_y = +2GM/(vb)$. It may seem, therefore, that the net result is a big fat nothing. However, in fact we have here an example of a random walk, where the "steps" are in velocity space. Even though there is an equal chance to increase as to decrease the velocity, the mean *square* of the velocity does change. Here's how it works. Suppose, in some arbitrary units, that the y component of the velocity is 1. An interaction changes this component to $1-\epsilon$ with a 50% probability, and to $1+\epsilon$ with a 50% probability. The average is still 1. However, the average square is $0.5(1-\epsilon)^2 + 0.5(1+\epsilon)^2$, or $1+\epsilon^2$. Thus, the square changes by ϵ^2. In our case of a two-body interaction, this means that

$$\langle(\Delta v_y)^2\rangle = \frac{4(GM)^2}{v^2 b^2} \ . \tag{5}$$

When the change in the mean square becomes comparable to v^2, the sum of all the little interactions has become significant. How long does this take?

If the star moves with speed v, then the number of stars with which it interacts in time dt with impact parameters between b and $b+db$ is the number density times the area times the distance traveled in time dt, or

$$dN = n 2\pi\, b\, db\, v\, dt \ . \tag{6}$$

Therefore, the change in $\langle v^2\rangle$ in time dt from that group of stars is

$$d\langle v^2\rangle = \frac{4(GM)^2}{v^2 b^2} n 2\pi\, b\, db\, v\, dt \ , \tag{7}$$

and the rate of change in $\langle v^2\rangle$ is given by integrating over impact parameters from the minimum $b_{\rm min}$ to the maximum $b_{\rm max}$, or

$$\frac{d\langle v^2\rangle}{dt} = \frac{4(GM)^2 2\pi}{v} n \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} = (8\pi/v)(GM)^2 n \ln(b_{\rm max}/b_{\rm min}) \ . \tag{8}$$

The *relaxation time* is the time necessary to change v^2 by of order itself:

$$\tau_{\rm rel} = v^2/(dE/dt) = \frac{v^3}{8\pi(GM)^2 n \ln(b_{\rm max}/b_{\rm min})} \ . \tag{9}$$

We haven't defined what the minimum and maximum impact parameters should be. Luckily, because they only appear in a logarithm, their exact value doesn't matter much. We can, in fact, use the tongue-in-cheek rule that "all logarithms are 10". What that means is that there is a huge range of values for which the logarithm is within a factor of 2 of 10: 100 to 10^9 fit the bill. If we wanted to get more precise, we could say that since we've assumed weak interactions, $b > GM/v^2$ (because otherwise the velocity would change by of order itself in a single interaction) and $b < r$ (since that's the size of the system). From virial equilibrium, $v^2 \sim GNM/r$, so $b_{\rm max}/b_{\rm min} \sim r/(GM/v^2) \sim r/(r/N) \sim N$, so $\ln(b_{\rm max}/b_{\rm min}) \sim \ln N$. The time to cross the system by free-fall is $t_{\rm cr} \sim r/v$, so we can write

$$\begin{aligned} \tau_{\rm rel} &\sim (GNM/r)(1/n)(v/(GM)^2)(1/\ln N) \\ &\sim (r/v)[(v/r)^2/(GMn)](N/\ln N) \\ &\sim t_{\rm cr}[GMN/(r^3 GMn)](N/\ln N) \\ &\sim t_{\rm cr} N/\ln N \ . \end{aligned} \tag{10}$$

Therefore, relaxation occurs in $\sim N/\ln N$ crossing times. A more exact calculation gives $\sim 0.1N/\ln N$ crossing times.

How long is this, for various systems? A globular cluster has $N \approx 10^6$ and $t_{\rm cr} \approx 10^5$ yr, so $\tau_{\rm rel} \sim 10^9$ yr. Globulars are $\sim 10^{10}$ yr old, so they are several relaxation times old and lots of evolution has happened. In contrast, a galaxy has $N \approx 10^{10}$ in the central bulge and $t_{\rm cr} \approx 10^8$ yr, so $\tau_{\rm rel} \sim 10^{16}$ yr, much longer than the age of the galaxy. Relaxation is not important for a galaxy, for the most part (unless there are centrally concentrated regions). Galaxy clusters have $N \approx 10^3$ and $t_{\rm cr} \approx 10^9$ yr, so $\tau_{\rm rel} \sim 10^{10}$ yr, comparable to their ages. Therefore, many galaxy clusters are expected to be at least partially relaxed.

All this is wonderful, but by now you may be wondering what the big honking deal is about relaxation. So what if the velocity vectors are scrambled a bit? The reason this is important has to do with a fundamentally important fact about gravity:

The less energy a self-gravitating system has, the faster it moves.

To see this, return to a one-body orbit. Heck, let it be a circle. The total energy is $E_{\rm tot} = W + K = -2K + K = -K$. Since $E_{\rm tot} < 0$ for a bound orbit, then the more negative it is, the larger (more positive) K is. As an example, Mercury moves faster in its orbit than Neptune does, and Mercury is deeper in the Sun's gravitational well.

Why is this important? Recall that we showed that if two equal-mass particles interact, one of which is moving faster than the other, then the faster-moving one gives up energy to the slower-moving one. That means that the faster-moving one loses energy. But loss of energy means that it sinks deeper into the cluster's potential, so it moves faster. This process continues, with the faster particles tending to lose energy and sink deeper, therefore moving faster and so on. **Ask class:** what does this mean about the evolution of the cluster as a whole? It means that even if all the particles are of equal mass and the cluster starts out with essentially uniform density, random interactions tend to grant extra speed (thus lower total energy) to some particles, which then sink to the center. Over a period of a few relaxation times, therefore, the center of the cluster gets denser and the outer part expands. This happens in such a way as to conserve total energy. Thus, the effective radius of the cluster increases with time. In the case of a globular, the outermost stars become susceptible to tidal stripping by the host galaxy.

We've talked about thermodynamic analogies, so here's another one. Although the energy is constant, the "entropy" of the system goes up because the outermost stars have a lot more volume to access than they did before (and this more than makes up for the smaller volume accessible to the core stars). The effective negative specific heat of gravity (take away energy and things move faster, i.e., get hotter), however, produces strange effects because heat flows from cold things to hot things. That takes some getting used to!

What if the stars in a cluster have a range of masses? Let us, briefly, return to the interactions between two stars. Suppose two stars approach each other on hyperbolic orbits. Let the first star have mass m_1 and the second have mass m_2. Assume that initially, both have speed v_0 as measured in the cluster frame and that both are moving in the direction $\hat{x}$. What is their speed in the cluster frame after they interact gravitationally?

As we did for stars of the same mass but different velocity, we simply need to take into account conservation of energy and linear momentum. Let the initial velocity of particle 1 be $\mathbf{v}_1 = v_0\hat{x}$, and the initial velocity of particle 2 be $\mathbf{v}_2 = -v_0\hat{x}$. **Ask class:** for masses m_1 and m_2, what is the velocity of the center of mass? We can get it by conservation of momentum:

$$\mathbf{v}_{\rm CM} = \frac{1}{M_{\rm tot}}\mathbf{p}_{\rm tot} = \frac{m_1 - m_2}{m_1 + m_2} v_0\hat{x} \; . \tag{11}$$

Therefore, in the center of mass frame, the initial velocities are

$$\begin{aligned} \mathbf{v}_{1,\rm CM} &= \mathbf{v}_1 - \mathbf{v}_{\rm CM} = 2m_2 v_0/(m_1+m_2)\hat{x} \\ \mathbf{v}_{2,\rm CM} &= \mathbf{v}_2 - \mathbf{v}_{\rm CM} = -2m_1 v_0/(m_1+m_2)\hat{x} \; . \end{aligned} \tag{12}$$

Say that the result of the interaction is that particle 1 is deflected by an angle θ, as measured in the center of mass frame. **Ask class:** by how much is particle 2 deflected? By θ, since otherwise momentum isn't conserved. Concentrate on particle 1. The new velocity in the center of mass frame is

$$\mathbf{v}'_{1,\mathrm{CM}} = |\mathbf{v}_{1,\mathrm{CM}}| \cos\theta \hat{x} + |\mathbf{v}_{1,\mathrm{CM}}| \sin\theta \hat{y}\ . \tag{13}$$

Adding back the center of mass velocity, the new velocity in the cluster frame is

$$\mathbf{v}'_1 = \frac{v_0}{m_1+m_2}\left(2m_2\cos\theta + m_1 - m_2\right)\hat{x} + \frac{v_0}{m_1+m_2} 2m_2 \sin\theta \hat{y}\ . \tag{14}$$

Therefore, the new speed is

$$\begin{aligned} v'_1 &= [v_0/(m_1+m_2)]\sqrt{m_1^2+m_2^2-2m_1m_2+4m_2^2+4m_2\cos\theta(m_1-m_2)} \\ &= [v_0/(m_1+m_2)]\sqrt{m_1^2+m_2^2+2m_1m_2+4[m_2^2(1-\cos\theta)-m_1m_2(1-\cos\theta)]} \\ &= [v_0/(m_1+m_2)]\sqrt{(m_1+m_2)^2+4m_2(m_2-m_1)(1-\cos\theta)}\ . \end{aligned} \tag{15}$$

Ask class: what does this say about how the speed changes? It says that if $m_2 > m_1$, star 1 gains speed from the encounter, but that if $m_1 > m_2$, star 1 loses speed. Therefore, the more massive star loses speed in an encounter of this type. Now let's check the expression. **Ask class:** what units, limits, or symmetries can we consider? The units are okay (we need a speed, and indeed the masses cancel out). The symmetries are okay (relabeling 1 to 2 gives the right result). At least two limits are also okay: $m_1 = m_2$ gives no change in energy, regardless of the angle, and $\theta = 0$ (no encounter) also doesn't change the speeds. We've only considered a restricted set of encounters (collinear, equal speed in the cluster frame), but this motivates what turns out to be a more general rule: interactions tend to produce faster-moving light objects and slower-moving heavy objects. In fact, remarkably, one finds that there is net zero energy transfer for isotropic interactions and a Maxwellian distribution of speeds when the average kinetic energy is the same for all masses. **Ask class:** what does that imply about the average speed as a function of mass? Since $\frac{1}{2}mv^2$ is constant, $v \propto m^{-1/2}$.

Given this result, **Ask class:** what does this imply about the relative locations of less and more massive objects as the cluster evolves? Since the more massive objects lose energy through interactions, they sink to the center and go faster. Indeed, this process of *mass segregation* is observed in many globulars, where in the center higher-mass stars can outnumber lower-mass stars, in contrast to the normal ordering. One also expects to find an excess of black holes and neutron stars in the centers of globulars, as well as binary stars (which count as two stars, mass-wise). This is likely to lead to a bunch of very interesting interactions, some of which I am investigating for their potential as gravitational wave sources.