The Geometry of the Universe

What does the constant k in the Friedmann equation really mean? In this lecture we will follow Chapter 4 of Liddle to show that it has close connections with the geometry of the universe. We will therefore start with a reminder of the properties of familiar flat (Euclidean) geometry before moving off to less familiar possibilities.

Flat Geometry

Remember high school geometry? Pretty simple and elegant, as compiled by Euclid more than two thousand years ago. In any plane in a three-dimensional flat space, any triangle has interior angles that sum to 180 degrees, independent of size. The ratio of the circumference to radius of any circle is 2π. If you take a vector and move it around so that at all times you keep its orientation fixed (this is called "parallel transport"), then when you return it to its original location it will always be in the same orientation as before. In addition, in a two-dimensional flat space if there are two points that are separated by dx in the x direction and by dy in the y direction (perpendicular to x), then the distance ds between those points is given by the Pythagorean theorem: $ds^2 = dx^2 + dy^2$. These properties follow from the infamous fifth postulate of Euclid: "If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles."

Urk! Stated more simply, this says that if two lines are parallel at one point, they never meet, otherwise they do. This is infamous because it's a lot more complicated than Euclid's other postulates and as a result mathematicians through the ages spent a lot of time trying to prove it from the first four. Instead, to the consternation of the community, Riemann showed that one could develop perfectly fine geometries in which the fifth postulate does not hold.

I've always been surprised that it took this long, because in two dimensions there is an everyday counterexample that would easily have been familiar to the Greeks: the surface of a sphere. That's what we'll treat next.

Spherical Geometry

Technically what we really mean is "sphere-like" in the sense that the surface is of "positive curvature". There are other surfaces with this property, but a sphere is a good thing to keep in mind so let's examine it a bit.

Suppose we confine ourselves to the surface of a sphere. This is therefore a two-dimensional surface. Suppose we draw a really big triangle, which has one apex at the

North Pole and the others at the equator, 90 degrees of longitude apart. It is easy to see that this triangle has three right angles, for a total of 270 degrees! How about a circle? Suppose we take the equator. Its circumference is $2\pi R$, where R is the radius of the sphere. You might be tempted to say that its radius is R, but remember that we are confined to the surface, so the actual radius of the circle *in the surface* is $(\pi/2)R$, so that the ratio of circumference to radius is 4! One can also show that parallel transport can lead to rotations of a vector.

Obviously, we have something different going on. Note, though (and this is really important for general relativity), that if you confine yourself to a very small portion of the sphere, the geometry is *almost* that of a flat plane. This is one reason that it actually took some pretty decent observation and arguments from the Greeks to realize that the Earth is closer to a sphere than to a plane. It isn't obvious from local observations (for example, I presume that you navigate the campus using a map designed for Euclidean flatness!).

However, you cannot map a plane to the surface of a sphere globally. That's why, e.g., Mercator projections make Greenland look so huge. There is, in fact, no coordinate transformation that makes the surface of a sphere look exactly like a plane everywhere. These are genuinely different geometries.

The universe, of course, has three spatial dimensions instead of the two on the surface of a sphere. We can, however, make the extrapolation to another dimension. If a hypersphere has four dimensions, its boundary will have three. To us, locally, such a boundary will look like a standard Euclidean space. Globally, however, it will be very different. Just as it is difficult for us to prove the Earth's sphericity, it is tough to probe the geometry of the universe from home. Nonetheless, our ability to observe very distant things has allowed us to place serious constraints on the geometry.

Hyperbolic Geometry

These are also called spaces of negative curvature. A commonly used example is a saddle, but something that is more representative is a hyperboloid, which is a hyperbola rotated around its minor axis. In two dimensions, the interior angles of any triangle add to less than 180 degrees and there are an infinite number of lines through some point P not on line L that never intersect L (as opposed to there being no such lines in a spherical geometry, and exactly one in flat geometry). As before, however, in small regions on a hyperboloidal surface, the geometry is close to flat. See Figure 1 for a representation of the three types of geometry.
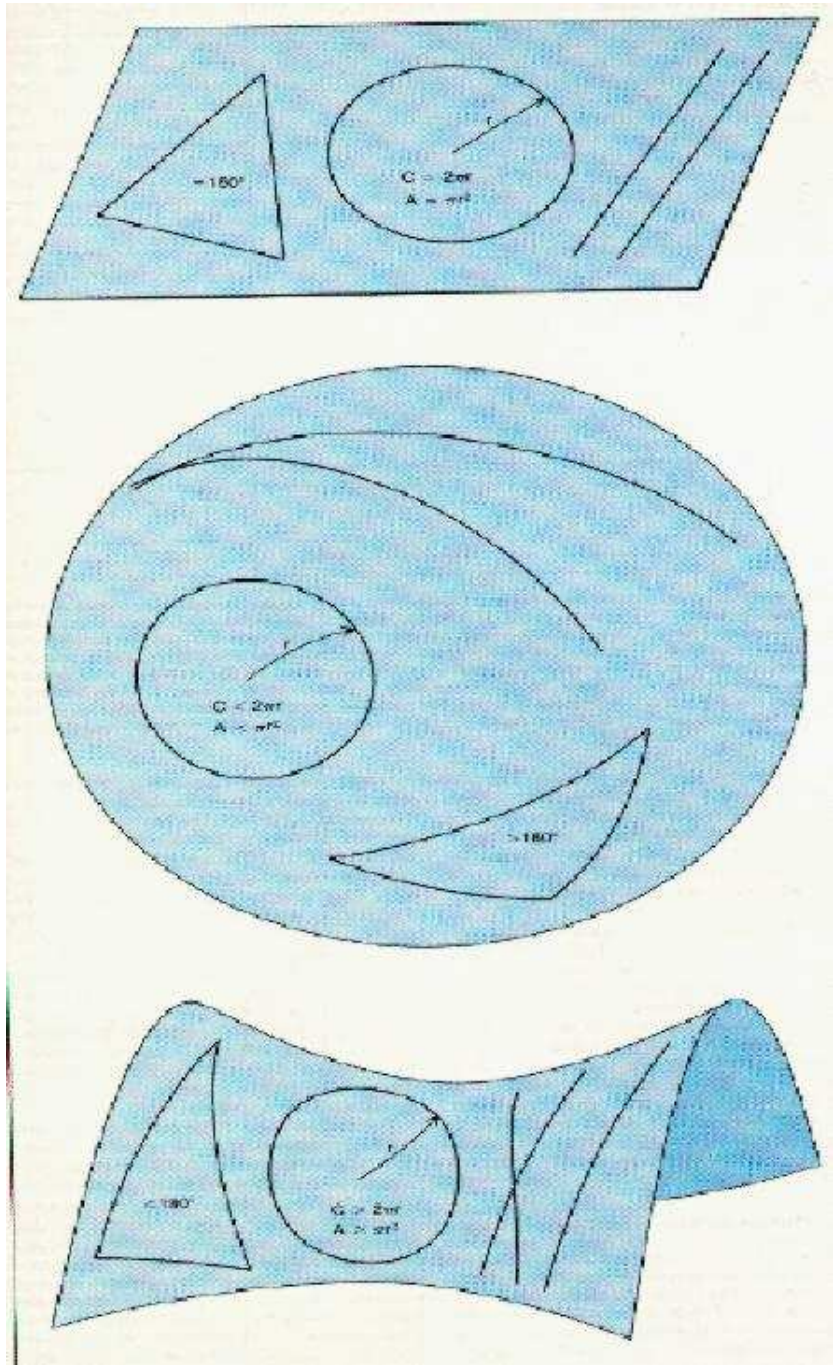
Finite or Infinite?

Fig. 1.— A representation of (from top) flat, closed, and open geometries. From http://zebu.uoregon.edu/∼imamura/123/images/topo3.jpg

One thing you might note about our examples is that for flat or hyperbolic geometry, the space is infinite, whereas for a spherical geometry the space is finite. As we will explore briefly in the next section this does not *have* to be the case, but it is in the simplest topology consistent with these geometries. What would this mean?

First, let's make something clear. In the standard interpretation, if the geometry is flat or hyperbolic, space is *and always has been* infinite. You should not, therefore, think of this as I used to, e.g., that a flat universe is currently finite but will expand forever. This might be a little difficult to imagine; how can something that is already infinite expand? Liddle suggests considering the positive integers: $1, 2, 3, \ldots$ When doubled, you get $2, 4, 6, \ldots$ but it's still infinite. Another way to think of it that might be closer to reality is to consider an infinite grid going in all directions. Now imagine that something pushes all the grid points farther from each other. Since the distance between any two given grid points increases, the grid expands, but is still infinite.

What's really bizarre, though is the implications of infinity. Various people (e.g., Max Tegmark) have had fun with this. Remember that we see only the part that is in causal contact with us, so the observable universe is finite no matter what. That means an infinite universe doesn't violate Olbers' paradox. However, if the universe is infinite then there are an infinite number of disconnected patches that are the same size and age as ours. If all the basic physical laws and basic constants are the same everywhere (e.g., all patches have the same number of macroscopic spatial dimensions and the same speed of light), then because our universe contains a finite number of particles that have interacted for a finite time, there would be an infinite number of absolutely exact copies of our observable universe, as well as all possible deviations however minor.

Let's drive that point home. That would mean that there is some universe out there identical in every way to ours except that Maryland had always beaten Duke in basketball when the two schools played. Sadly, there would also be a universe where the reverse was true. That kind of gives me the willies.

Even weirder is the possibility that these other patches have different values of physical constants, or different physical laws altogether. This has been suggested as one solution to the anthropic (i.e., human-centered) principle, which is basically the question of why the universe can support life. If almost all universes can't, then it is no surprise that in one that can we note that we can exist. This would still leave the question of what happens at the boundary between universes with different constants or laws.

What can be said observationally? As we will discuss later in more detail, evidence from the CMB and elsewhere is currently consistent with the universe having a flat geometry. Note, though, that the only thing that we can ever really say is that the cosmos (i.e., all there is, whether observable by us or not) is much bigger than the patch we see. It could be,

for example, that the cosmos is actually closed and finite but we happen to be in a slightly underdense region that looks open or flat.

It is, in any case, fun to speculate about this stuff as long as your brain doesn't explode.

Topology

To aid in the explosion, though, let's draw a distinction between *geometry* (the local properties of spacetime) and *topology* (the global properties). Suppose that the universe has a flat geometry. It could, for example, actually be a torus, which from the topological standpoint is equivalent to taking a square and identifying opposite sides with each other (think of a game in which a character moving off one edge appears on the other). Triangles on a torus add up to 180° and otherwise Euclid is happy. We therefore must be careful not to draw too many sweeping conclusions about the cosmos from some local measurements! In principle, if there is such periodicity in the universe, one would expect to see evidence of it from repetition in CMB patterns at widely spaced points in the sky, but nothing of the sort has been seen.

Where did the Big Bang Happen?

Let us stress once again that regardless of the geometry or topology, we stick to our cosmological principles of homogeneity and isotropy. Among other things, this means that you cannot point to any location in the universe and say "This was the unique spot that was the center, where the Big Bang happened." Consider the standard balloon analogy. You blow up a balloon, and it gets bigger and bigger. Which specific spot on the balloon was where the expansion started? In a mathematically perfect balloon that started at a point, the answer is that all points came from that, so none has a special place. One aspect of this analogy that may seem confusing is that, of course, there actually is a center to the balloon (right in the middle, where air is). The counterpart in this case is that one *can* say that there was a unique *time* when the Big Bang happened, just not a unique *place*. This is important to keep in mind.

The meaning of k

Remember the Friedmann equation? It had a mysterious constant k:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{1}$$

where $kc^2 \equiv -2U/mx^2$ and U was a conserved energy $\propto x^2$. When $k \neq 0$ one often rescales the scale factor via $\hat{a} = a/\sqrt{|k|}$, so that (with $c \equiv 1$) the equation becomes

$$\left(\frac{\dot{\hat{a}}}{\hat{a}}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\hat{k}}{\hat{a}^2} \tag{2}$$

where $\hat{k}$ (my usage; not standard) is 0 +1, or −1 depending on whether the universe is flat, closed, or open, respectively. The last term is therefore related to the geometry of the universe.

Intuition Builder

You know that you can make a flat map out of any small portion of a sphere (e.g., the Earth). Can you convince yourself, though, that you cannot project the *entire* sphere onto a plane? We're not worried here about distortions, just whether every point on a sphere can be uniquely mapped onto a plane.