Simple Cosmological Models

Following Liddle's Chapter 5, we will now explore some simple solutions to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1}$$

and the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0\ . \tag{2}$$

A straw poll of cosmologists roughly a decade ago would have found most of them agreeing that these equations describe the universe pretty well. However, since that time strong evidence has emerged that the universe is accelerating in its expansion. This requires an additional ingredient, called dark energy.

For now, though, let's ignore that and go back to the blissful days before dark energy. What is the left hand side in the Friedmann equation? Note that if we consider two objects that are currently at a distance r from each other and are moving with the universal expansion, then at any given time their separation is proportional to the scale factor: $r \propto a$, hence $\dot{r} \propto \dot{a}$. Therefore, $\dot{r}/r = \dot{a}/a$. We note from the Friedmann equation that as a result, $\dot{r}/r$ is independent of r. This is just what Hubble's Law tells us: the apparent recession speed is proportional to the distance. We can define $H \equiv \dot{a}/a$, and H_0 as the value right now. Note that, indeed, the units of the Hubble "constant" are one over time: $H_0 = 72$ km s^{-1} Mpc^{-1} (distance per time per distance). We therefore find an evolution equation for $H = H(t)$:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\ . \tag{3}$$

Let's consider what this means. We know that ρ decreases as a increases. If $k = 0$ then as a increases H therefore also decreases, so the expansion slows down with time. If $k < 0$ then both terms are positive and both terms decrease as a decreases, so again H decreases with increasing a and thus the expansion slows down. If $k > 0$ then since $H > 0$ now we know the first term on the right hand side is larger than the second. However, for $\rho \propto 1/a^3$ (for nonrelativistic matter) or $\rho \propto 1/a^4$ (for relativistic matter), the first term decreases more rapidly with increasing a than the second. As a result, when $k > 0$ there will come a time when $H = 0$, at which point the expansion has stopped and the universe will recontract. Even in this case, though, since $H > 0$ now we expect the expansion to slow down in the future. As a result, without something else happening, an accelerating expansion is not expected.

Expansion and Redshift

What happens to a photon as the universe expands? The answer, in short, is that the wavelength of the photon is proportional to the scale factor a (see Figure 1). As Liddle shows, the straightforward way to motivate this is to consider two nearby galaxies, with separation dr. Assuming that they both move with the universal expansion, this tells us that their apparent relative speed is $dv = (\dot{a}/a)dr$. Light is emitted from one at wavelength λ_e, and the Doppler shift is therefore given by

$$d\lambda/\lambda_e = dv/c \; . \tag{4}$$

We also know that it takes a time $dt = dr/c$ to travel the distance, so that

$$\frac{d\lambda}{\lambda_e} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a} \; . \tag{5}$$

Integrating gives $\lambda \propto a$. You can then imagine a large number of such infinitesimal motions, calculus-style, and conclude that $\lambda \propto a$ is valid for any distance traveled. The redshift z is usually defined as

$$1 + z = \frac{\lambda_r}{\lambda_e} = \frac{a(t_r)}{a(t_e)} \tag{6}$$

where a photon emitted at wavelength λ_e is received at wavelength λ_r, and the global time of emission and reception are t_e and t_r, respectively. This definition implies $z = 0$ at the present time. The redshift (or scale factor) is the most relevant single quantity to refer to cosmological epochs. Unfortunately, the popular press likes to talk about *times* instead: "the most distant galaxy was reported yesterday, having emitted the light we see 13 billion years ago." The problem is that quoting a time like that requires a specific cosmological model, whereas the redshift is a statement of observational fact!

Solving the Equations

The fluid equation involves both the mass-energy density ρ and the pressure p. As a result, to follow the evolution of the universe we need to know the relation $p(\rho)$ (or more generally when the temperature is important, $p(\rho, T)$). This is called the *equation of state*. This comes up in many contexts in physics. For example, consider water. When it is water vapor, a small change in pressure leads to a significant change in density. In contrast, when it is liquid, a very large change in pressure is needed for even a small change in density. This tells us that in the most general circumstances we need to worry about *phase changes*, where a substance alters its character due to a change in density, temperature, or something else. These can have importance in the early universe, where, for example, it is thought that there was a shift from a very high-density high-temperature environment in which quarks

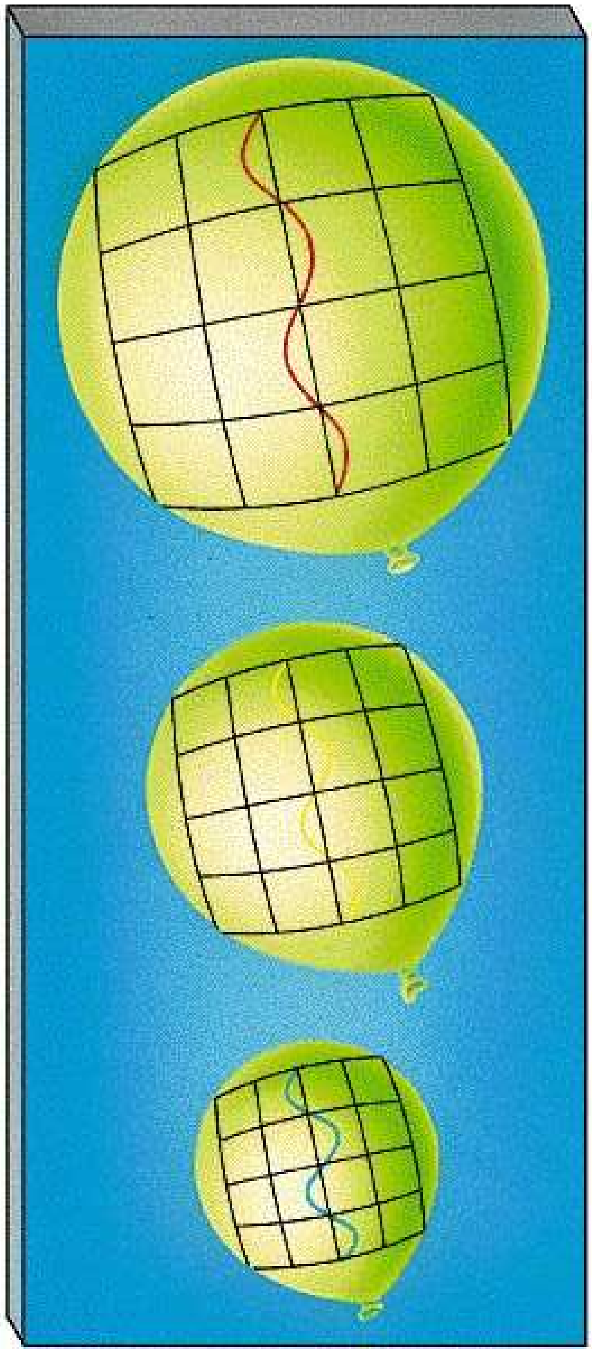

Fig. 1.— Cosmological redshift. As the universe expands, the wavelength of a freely moving photon increases proportional to the scale factor. Therefore, the redshift is not really due to motion per se, but to the scale of the universe. From http://www.pas.rochester.edu/∼afrank/A105/LectureXVI/FG26_007.JPG

and gluons were free, to the current lower-density lower-temperature environment in which the quarks and gluons are bound in hadrons. Near such density-induced phase transitions, matter tends to be more "squeezable" (less pressure change for a given density change), and one cool consequence is that some people think that this phase of the universe was ripe for the production of primordial black holes.

For our initial attempt at the equations, though, we will consider only two simplified equations of state. One of them is relevant for nonrelativistic matter. For such matter, our approximation is that $p \approx 0$, which is really to say that $p/c^2 \ll \rho$. The other is completely relativistic radiation. For this, we can derive the equation of state as follows.

Suppose that we have radiation with a number density n and an energy per photon of E. The momentum is then $P = E/c$. Say that we put the radiation in a box, and ask about the pressure that is exerted on a wall of area A. The pressure is the force per area, and the force is the momentum per time. The momentum per time is the number of photons per time, times the momentum per photon. The number of photons per time in a given direction is nAv, where $v = c/\sqrt{3}$ if we assume the radiation is moving isotropically (to see this, note that in each of three orthogonal directions $v = c/\sqrt{3}$, hence the total speed is $v^2 = 3(c/\sqrt{3})^2 = c^2$). If the total energy of the photon is E then the momentum in a given direction is, similarly, $E/\sqrt{3}c$. Then the force is $nAc/\sqrt{3} \times E/\sqrt{3}c = nAE/3$, and the pressure is $p = nE/3$. The energy density is $\rho c^2 = nE$, so the equation of state is just $p = \rho c^2/3$ or $p = \rho/3$ in units with $c \equiv 1$.

Starting, then, with nonrelativistic matter, the fluid equation is just

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \,, \qquad \rho \propto a^{-3} \,. \tag{7}$$

This makes sense! A whole bunch of particles passively moving with the universe has a number density that goes as a^{-3}. Now suppose that $k = 0$ and that we set $a \equiv 1$ at the present time, with $\rho = \rho_0$ now. The Friedmann equation becomes

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a} \,. \tag{8}$$

Faced with an equation like this, a power law is a good guess, and indeed we find that

$$a(t) = (t/t_0)^{2/3} \,; \qquad \rho(t) = \rho_0(t_0/t)^2 \,; \qquad H \equiv \dot{a}/a = 2/(3t) \,. \tag{9}$$

Note, therefore, that the current age of the universe can be obtained simply: $t_0 = 2/(3H_0)$. Doing this with the best value of H_0 (72 km s^{-1} Mpc^{-1}) gives about 9 Gyr (i.e., 9 billion years). The oldest stars are estimated by various techniques to be about 12 Gyr old. That is a problem that was starting to mount its head in the 1980s and 1990s.

With photons, we find $\dot{\rho} + 4(\dot{a}/a)\rho = 0$, meaning that $\rho \propto a^{-4}$. Again, this makes sense. The number density of passively evolving photons goes as a^{-3}, but the energy of each photon is also redshifted as $E \propto a^{-1}$, giving a mass-energy density $\propto a^{-4}$. We then find

$$a(t) = (t/t_0)^{1/2} \; ; \qquad \rho(t) = \rho_0 (t_0/t)^2 \; . \tag{10}$$

This also implies that in a radiation-dominated epoch, $H = 1/(2t)$. Note that the universe expands more *slowly* than during the matter-dominated epoch. Pressure is a form of energy, and like any energy it gravitates, so again one should be cautious not to think of pressure as blowing up the universe.

The current universe is overwhelmingly matter-dominated, but has that always been the case? The answer is no, as can be seen by $\rho_{\rm rad} \propto a^{-4}$ but $\rho_{\rm matter} \propto a^{-3}$. There was a stage, at a redshift of a few thousand and above, when radiation dominated. Therefore, the early part of the expansion was driven by radiation, but more recently has been governed by matter.

What About Curvature?

We obtained the previous solutions by assuming $k = 0$, but perhaps the universe doesn't work like that. What if $k \neq 0$?

Consider again the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \; . \tag{11}$$

If $k \neq 0$ we might worry that the whole nature of the solutions changes. However, we can obtain substantial insight by assuming that either the first term or the curvature term (the one with k) dominates. Note that for matter $\rho \propto a^{-3}$ and for radiation $\rho \propto a^{-4}$, whereas the curvature term is $\propto a^{-2}$. This tells us right away that when a is small enough, the first term is more important. In such early phases, we really can ignore the curvature term for our solutions, and since measurements indicate that the curvature term is currently small (and possibly zero), the entire history of the universe thus far has had minimal influence from curvature. What about later, though? If $k < 0$ then the universe will expand forever, meaning that a will grow arbitrarily large. It is therefore inevitable that the curvature term will dominate, at which stage the Friedmann equation becomes $(\dot{a}/a)^2 \propto 1/a^2$, or $\dot{a} \propto$const, meaning $a \propto t$. The universe thus enters a coasting phase. If $k > 0$ then eventually the right hand side becomes zero, the universe turns around, and recollapses.

We therefore conclude that unless $k = 0$, curvature will eventually dominate. Note, though, that this does *not* apply if there is dark energy around!

Intuition Builder

Suppose the main component of the universe is some sort of field with an equation of state $p = w\rho$ (with $c \equiv 1$). What are the requirements on w for this field to be more important than curvature at very late times (i.e., large a)?