The Cosmological Constant

No more teasing: we're finally here! In this lecture we will introduce the cosmological constant (and some more general versions of dark energy) and discuss a few of its implications.

Early Motivation

After Einstein introduced the final version of his field equations in November 1915, people got to work on them and discovered fairly quickly that they implied that the universe as a whole is dynamic (either expanding or contracting). However, aesthetic biases and the astronomical observations of the time suggested that the universe was actually static as a whole. As a result, Einstein introduced a new term into his equations: the cosmological constant Λ. This comes into the Friedmann equation as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \Lambda/3 \tag{1}$$

and into the acceleration equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p/c^2\right) + \Lambda/3 \; . \tag{2}$$

Note that Λ has units of time^{-2} in these equations. Note also that it is supposed to be a genuine *constant*, independent of time or space.

What does this mean? If all you wanted was that the universe happened not to be expanding or contracting *at this moment*, then your only requirement is that $\dot{a} = 0$, which you could manage with $\Lambda = 0$. However, that's like saying that if you throw a rock straight up into the air, there will be a moment when it isn't moving; you know it will come down soon!

To have a truly static universe, you need $\dot{a} = 0$ and $\ddot{a} = 0$. Assuming a pressureless universe $3p/c^2 \ll \rho$, $\ddot{a} = 0$ gives $\Lambda = 4\pi G\rho$. Substituting this into the Friedmann equation with $H = 0$ then gives $k/a^2 = 12\pi G\rho/3$. Therefore, only a positive Λ and a positive k (i.e., a closed universe) can give Einstein's static solution.

The problem, though, is that this solution is unstable to small perturbations. Look at the acceleration equation: the matter term decreases the expansion speed with time (negative $\ddot{a}$), whereas a positive Λ increases the expansion speed with time. Suppose you have the static balance. Now imagine that the universe gives a little burp and increases a slightly. The density goes down as $\rho \propto a^{-3}$, but Λ stays constant. As a result, $\ddot{a} > 0$, leading to $\dot{a} > 0$. This further expansion of the universe increases $\ddot{a}$ yet more, and the universe just keeps on expanding. If the fluctuation had decreased a slightly, it would collapse. This is akin to balancing a sharpened pencil on its tip: maybe you could do it, but any tiny fluctuation (e.g., a stray air molecule) would cause it to deviate exponentially from that state. Therefore, Λ does not do what Einstein wanted it to.

When Edwin Hubble established in 1929 that the universe is in fact expanding, Einstein bitterly regretted adding the cosmological constant to his theory, calling it "the greatest mistake of my life." Would that I could make such a mistake! Λ is making a comeback based on the observations we have discussed.

To put the cosmological constant on the same footing as other terms, one commonly defines $\Omega_\Lambda \equiv \Lambda/(3H^2)$, in addition to $\Omega_k \equiv -k/(a^2H^2)$. The Friedmann equation then becomes simply

$$\Omega + \Omega_k + \Omega_\Lambda = 1 \; . \tag{3}$$

It is important to note that although this equation always holds, the three components have different dependences on a and therefore the value of any one of them is not constant with time. We'll get to that more a bit further on.

Fluid description

We can abuse our notation a bit more by defining

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \tag{4}$$

to put the Friedmann equation into the form

$$H^2 = \frac{8\pi G}{3}\left(\rho + \rho_\Lambda\right) - k/a^2 \; . \tag{5}$$

This means that $\Omega_\Lambda = \rho_\Lambda/\rho_c$. It also means that the fluid equation for Λ becomes

$$\dot{\rho}_\Lambda + 3\frac{\dot{a}}{a}\left(\rho_\Lambda + p_\Lambda/c^2\right) = 0 \; . \tag{6}$$

To be a true constant this means

$$p_\Lambda = -\rho_\Lambda c^2 \; . \tag{7}$$

Whazzat??? How on earth could the pressure for anything be negative? Amazingly, particle physicists have a ready-made answer for that, and even some experiments to back them up.

Quantum mechanics predicts that the vacuum isn't really empty. Instead, it is chock full of "zero point energy." Remember the quantum harmonic oscillator? The energy of the oscillator can be $(n + 1/2)\hbar\omega$ for some frequency ω, with $n = 0, 1, 2, \ldots$, but this means that even the ground state $n = 0$ has an energy $\hbar\omega/2$. For nongravitational purposes only differences in energy are meaningful, but gravity is supposed to couple to all forms of energy. That means that this energy should matter, summed over all frequencies ω. Examination of the effect indicates that it could well give us the ρ_Λ we need.

In addition, it actually does give us a negative pressure. Experimental confirmation of this comes from something called the Casimir effect: if you put two uncharged parallel metal plates close to each other in a vacuum, they feel an attractive force that is produced by an effectively negative pressure between them. The physical idea behind this is that although modes of all wavelengths can exist outside the plates, in the region between them only some can exist (the ones whose wavelengths fit into the region), hence there is more pressure outside than in and the plates go together. There is even a real-world analogy that I haven't personally observed but am told can be seen. Suppose you have two ships next to each other (with some separation), just floating along and initially not in relative motion. They will gradually come together. As in the Casimir effect, the waves between the boats are limited in wavelength, but outside are not, so the pressure between the boats is decreased by comparison.

Great! Sounds like we're basically home free. As a quick check, though, we should probably make a rough estimate of the magnitude of the effective density ρ_Λ that we expect. If you ask a particle physicist to do this you'll probably hear something like (I'm paraphrasing): "mumble mumble quantized modes mumble ultraviolet divergence mumble Planck cutoff mumble mumble." As it turns out, though, we can make an estimate on dimensional grounds that is about right to within paltry factors such as $16\pi^2$. The expected density ρ_Λ has to be constructed out of the basic constants G, $\hbar$, and c. The only such combination is

$$\rho_\Lambda(\text{expected}) \sim \frac{c^5}{\hbar G^2} \approx 5 \times 10^{96}\ \text{kg m}^{-3}\ . \tag{8}$$

The present best estimate for the actual value is

$$\rho_\Lambda(\text{measured}) \sim 0.73\rho_{\text{crit}} \approx 7 \times 10^{-27}\ \text{kg m}^{-3}\ . \tag{9}$$

Even by the standards of cosmology, this is a pretty hefty error! A lot of work has gone into trying to figure out whether there are natural effects that would cancel most of this out, but 120-odd orders of magnitude is rather substantial and at this point there is no particular agreement on what the answer might be. If ρ_Λ were *exactly* zero that might be attributable to some perfect cancellation, but for it to be cancelled out to this degree yet be big enough to have an impact now? Egad.

Cosmological models with Λ

If $\Lambda \neq 0$, how does the universe evolve? Thinking about the Friedmann equation, remember that:

- The radiation term (or relativistic matter) scales as a^{-4}.
- The pressureless matter term scales as a^{-3}.
- The curvature term scales as a^{-2}.
- The cosmological constant term scales as a^{0}.

As a result, we are guaranteed that radiation and relativistic matter will dominate for sufficiently small a and that any cosmological constant will dominate for sufficiently large a.

It therefore turns out that at large scale factors some of our previous conclusions about the relation between geometry and the fate of the universe need not apply. For example, we had previously said that an open universe (for which $\Omega_k > 0$) would expand forever. However, consider a universe in which $\Omega_k > 0$ but $\Omega_\Lambda < 0$. The cosmological constant then acts as an extra *attractive* force. As the universe expands this will eventually take over, reverse the expansion, and cause a collapse. Similarly, you could arrange parameters such that $\Omega_k < 0$ yet the universe expands forever because a positive Ω_Λ takes charge. It is still the case, though, that an open or flat universe is spatially infinite.

What if we are in a phase in which Ω_Λ is the only important term? In that case the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 \approx \Lambda/3 \tag{10}$$

with the solution $a \propto \exp(t\sqrt{\Lambda/3}) = \exp(Ht)$. That is, this produces an exponentially expanding universe.

We can also generalize further to equations of state of the form $p = w\rho c^2$, where w need not even be a constant with time. We call this "dark energy". If w is constant and $w > -1$, the evolution equations imply

$$\begin{aligned} \rho(a) &\propto a^{-3(w+1)} \\ a(t) &\propto t^{2/3(w+1)} \\ \rho(t) &\propto t^{-2} \ . \end{aligned} \tag{11}$$

If $w < -1/3$ then this component will dominate the evolution of the universe at sufficiently large a. See Figure 1 for some past and future implications of a cosmological constant.

Dicke Coincidences

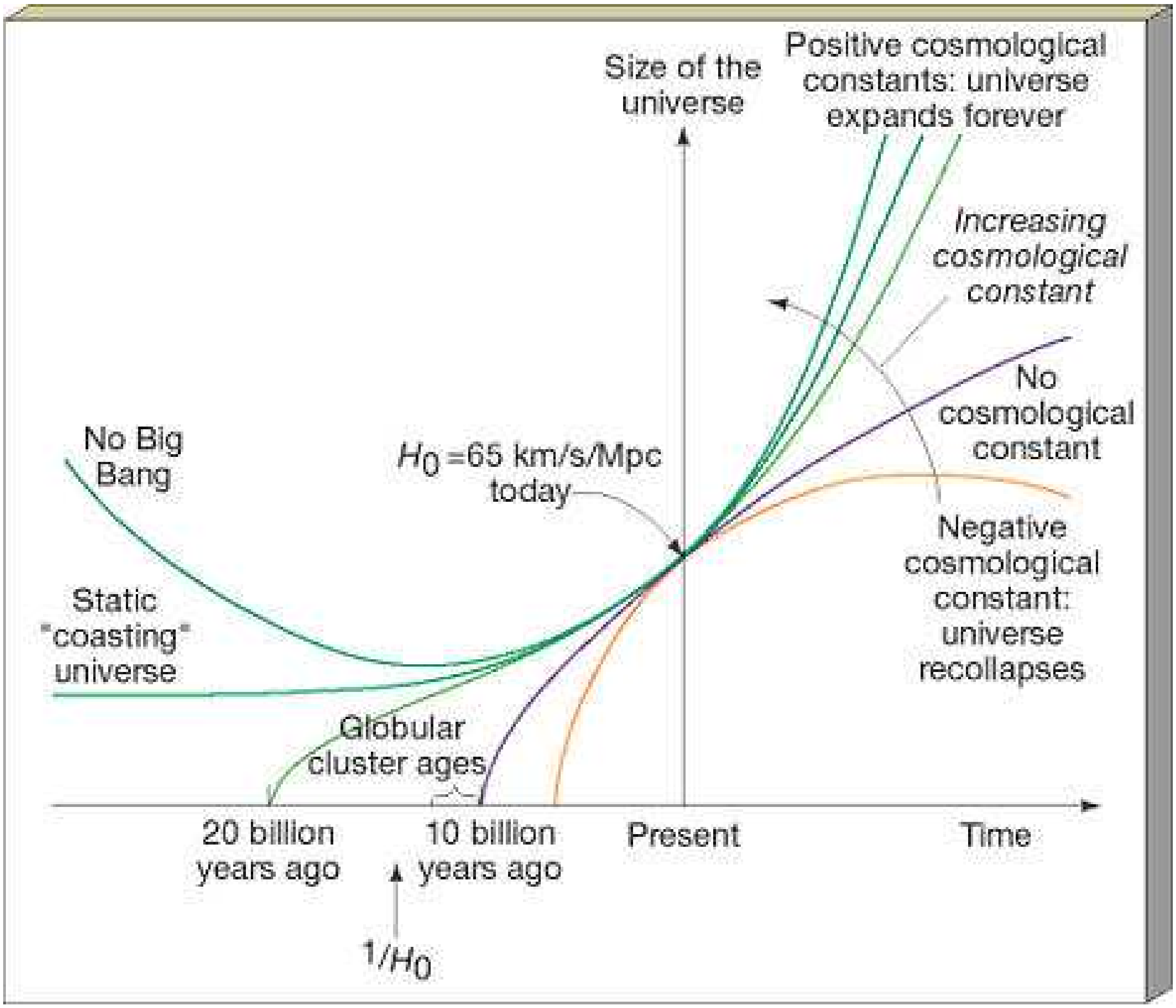

Fig. 1.— Effects of a cosmological constant on the evolution of the universe. From http://physics.uoregon.edu/∼courses/BrauImages/Chap27/IN27_101.jpg

There are some philosophical implications of dark energy that were first stressed by Robert Dicke in 1970. Note that Ω, Ω_k, and Ω_Λ all scale differently with a. Note also that a has gone through many orders of magnitude of evolution. That means that during almost all of that time (in a logarithmic sense), and in the future as well, only one of the three can be remotely significant. Dicke thus argued that it would be really amazing and anti-Copernican if, just at our epoch, more than one of the three is measurably nonzero.

What, then, can we make of $\Omega = 0.27$ and $\Omega_\Lambda = 0.73$, which are the best current values? There is as yet no satisfactory answer. A lot of people appeal to an anthropic argument; we don't want to be silly and announce that the presence of *humans* in particular is of any consequence, but we can note that only a universe with life could produce creatures who wonder about it, so our presence in a universe that can support life is not surprising! If different universes have different values of Λ, it might therefore be reasonable that ones with very much larger Λ don't produce life, because those universes blow up before any matter can get together to form complex entities such as stars and planets. It is, however, not at all clear why a lower Λ would be a problem, or why in particular it is essential to have a value of Λ that has significant impact on the universe at around the same time that stars and galaxies form.

Maybe we're making too much of this. As another example, note that the angular size of the Moon from Earth is very close to that of the Sun, meaning that total eclipses can happen but they are so rare in any given location that they cause great awe. The Moon is drifting away because of tidal forces, meaning that for most of Earth's history this close equality didn't exist. Is our appearance on the world stage at this special time just happenstance? Well, yes! Going back to Λ, I think it's fair to say that we won't be able to make intelligent comments on the level of coincidence until particle physicists have a handle on its origin, and therefore on the range of its possible values.

Intuition Builder

The theoretical calculation of ρ_Λ we did involves a cutoff energy beyond which zero-point energy does not exist. What if, for some reason, that cutoff energy is much lower than we assumed? To get ρ_Λ to the observed range would require a cutoff energy of about 0.01 electron volts. Is this possible, or are there experiments that might rule out such a low cutoff?