The Age of the Universe

One somewhat low-profile but important test of cosmological models is that the age of the universe implied by the models agrees with the age estimated using various methods. Here we'll talk about the age we get from models, and in the next lecture will discuss several observational handles on the true age.

For a change of pace, and to give us some practice in evaluating complicated equations, we will start with the equation for the *lookback time* provided by Hogg in astro-ph/9905116, and see whether the units, limits, etc. make sense. The lookback time from now ($z = 0$) back to redshift z means the time that would have elapsed on the wristwatch of an observer moving with the Hubble flow between those redshifts. Hogg's expression is:

$$t_L = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\sqrt{\Omega(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\Lambda}} \; . \tag{1}$$

Here H_0 is the current value of the Hubble parameter, and Ω, Ω_k, and Ω_Λ each indicate their *current* ($z = 0$) values. Does this expression behave as it should? See Figure 1 for a graphical representation.

As always, we check units first. Recall that the Hubble parameter has units of inverse time, so the prefactor $1/H_0$ has units of time. The integral is dimensionless because the redshift is, so indeed the units are correct.

Our second easy check is to note that t_L must increase with increasing z. Since the integrand is always positive, this is indeed the case.

Our third simple check is to note that smaller H_0 should imply a larger time back to a given redshift, because $H_0 = \dot{a}/a$ today. That's another motivation for the $1/H_0$ factor in front.

What's the next easiest thing to do? Suppose we have a universe in which $\Omega_\Lambda = 1$ and $\Omega = \Omega_k = 0$. Then the lookback time is

$$t_L = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\Omega_\Lambda^{1/2}} \; . \tag{2}$$

Since $\Omega_\Lambda = 1$, this says that $t_L = \ln(1+z)/H_0$. But hold on! This says that if you go to an arbitrarily large redshift z, you get to an arbitrarily large lookback time t_L. This means that either the universe is infinitely old, or it started at some finite scale factor and therefore z cannot get arbitrarily large. Did Hogg make a mistake?

This requires careful examination on our part. First, note that if the current scale factor is a_0, this says that since $1 + z = a_0/a$, the solution is $a = a_0 \exp(-H_0 t_L)$. This is in fact the solution we obtained previously, so it at least is consistent. The more important physical

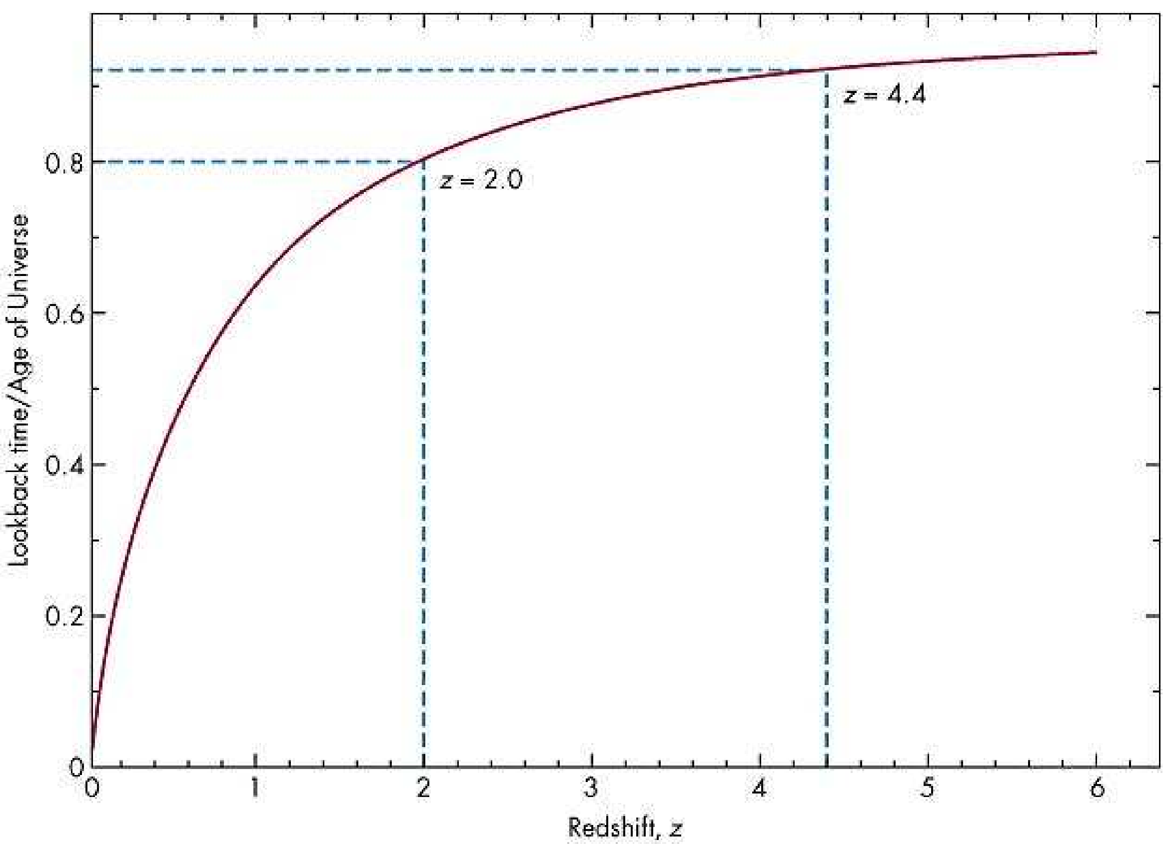

Fig. 1.— Example of lookback time versus redshift. From http://www.mhhe.com/physsci/astronomy/fix/student/images/24ex3.jpg

insight is to realize that because the matter contribution scales as $(1+z)^3$, at high enough redshift it will dominate (since we know $\Omega > 0$!), hence it is meaningless to continue the exponential solution too far back into the past. In fact, this is a good general point to make. Currently the best estimate is that $\Omega = 0.27$, $\Omega_\Lambda = 0.73$, and $|\Omega_k| < 0.03$. At a redshift such that $1 + z = 10$ (around the time of formation of the first galaxies), $|\Omega_k| < 0.003$, $\Omega_\Lambda \approx 0.0007$, and $\Omega \approx 0.997 - 1.003$. Even when $1 + z = 2$, we had $|\Omega_k| < 0.01$, $\Omega_\Lambda = 0.09$, and $\Omega \approx 0.9$. This tells us that for $z > 1$, we are pretty well justified in using $\Omega \approx 1$ and $\Omega_k \approx \Omega_\Lambda \approx 0$. Therefore, although the cosmological constant apparently plays a major role now, it was negligible in the early universe, except during the extremely early inflationary epoch (which might have physics related to current dark energy).

With that in mind, we can move on to the case in which $\Omega = 1$ and $\Omega_k = \Omega_\Lambda = 0$. The lookback time then becomes

$$t_L = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')^{5/2}} = \frac{2}{3H_0}\left(1+z'\right)^{-3/2}\Big|_z^0 . \quad (3)$$

Specifically, this means that the full age of the universe since the $z \to \infty$ Big Bang is $T = 2/(3H_0)$.

Let us compare this to the case of an empty universe, with $\Omega = \Omega_\Lambda = 0$ and $\Omega_k = 1$. In that case we have

$$t_L = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')^{2}} = \frac{1}{H_0}\left(1+z'\right)^{-1}\Big|_z^0 . \quad (4)$$

In this case, the full age of the universe since the Big Bang is $T = 1/H_0$, so it is older than in the case of a flat matter-dominated universe. Is this reasonable?

To evaluate this, think about what we are fixing in the comparison. In both cases, we assume the same *current* value of the Hubble parameter, H_0. In the $\Omega = 0$ case, there is nothing to slow this down, meaning that the universe has been expanding at the same "speed" the entire time. In contrast, in the $\Omega = 1$ case, gravity has been putting on the brakes, meaning that the expansion was faster in the past. The average "speed" has thus been greater, meaning that the time to get where we are now has been shorter. It therefore does make sense. In a similar way, since the expansion rate is currently accelerating, it means that in the recent past the expansion was *slower* than it would have been in an $\Omega = 1$ or even a $\Omega_k = 1$ universe. The accelerating expansion therefore has the somewhat counterintuitive effect of implying a *larger* current age for the universe than we would infer otherwise. This will turn out to be important in reconciling the model age with the age inferred from various observables.

What is the overall scale of this age? We can define the Hubble time as

$$t_H \equiv 1/H_0 = 13.6 \text{ Gyr} \quad (5)$$

where as usual we used $H_0 = 72$ km s^{-1} Mpc^{-1} and to be cool we have written this in terms of Gyr= 10^9 yr units. This doesn't seem too bad. Even for a $\Omega = 1$ universe, this gives an age of 9 Gyr. That is comfortably larger than the age of the Earth. However, it is *not* larger than the ages of the oldest stars, which are thought to be more like $12 - 13$ Gyr old. Since it is generally accepted that the universe should be older than its oldest stars, this would be a problem! This was another motivation, in the 1990s, for considering a cosmological constant.

Age from the Friedmann Equation

We can also check Hogg's expression by solving the Friedmann equation directly. First consider $\Omega = 1$ and $\Omega_k = \Omega_\Lambda = 0$. The equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \; . \tag{6}$$

We saw earlier that this implies $a \propto (t/t_0)^{2/3}$, meaning that $H = (\dot{a}/a) = 2/(3t)$, and in particular $H_0 = 2/(3t_0)$ where t_0 is the current age of the universe. Rearranging gives us the result from Hogg's formula, $t_0 = 2/(3H_0)$. This check therefore works.

If instead $\Omega_k = 1$ and $\Omega = \Omega_\Lambda = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = -k/a^2 \; . \tag{7}$$

With $k = -1$, this means $\dot{a} \propto$const, so that $a \propto t$. Therefore $H = \dot{a}/a = 1/t$, $H_0 = 1/t_0$, and $t_0 = 1/H_0$, just as Hogg's formula gives. Earlier we also showed that when $\Omega_\Lambda = 1$ and $\Omega = \Omega_k = 0$ the formula works as well, so in the simple limits everything is okay.

Other Limiting Cases

If you have a lot of time on your hands and/or access to a symbolic manipulation program, you can demonstrate that there are exact solutions to a couple of slightly more general cases. The most relevant one is the case with $\Omega_k = 0$ but current values of $\Omega = \Omega_0$ and $\Omega_\Lambda = 1 - \Omega_0$ that can be anywhere between 0 and 1. Painful integration then yields

$$H_0 t_0 = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}} \ln\left[\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right] \; . \tag{8}$$

For practice, let us close up by applying our usual tests to see if this expression is reasonable.

First, we see that the units work out: H_0 has units of inverse time, t_0 has units of time, and the right hand side is unitless.

Second, what if $\Omega_0 = 1$, where we expect $t_0 = 2/(3H_0)$? We see an apparent problem: we have a 0/0 (the log is zero, as is the first square root). We therefore have to use L'Hopital's rule (somewhat complicated) or take a limit. Let $\Omega_0 = 1 - \epsilon$, with $\epsilon \ll 1$. Then the log becomes

$$\ln \left[\frac{1 + \sqrt{\epsilon}}{\sqrt{1 - \epsilon}} \right] \approx \ln(1 + \sqrt{\epsilon}) \approx \sqrt{\epsilon} \tag{9}$$

and the square root becomes $\sqrt{\epsilon}$. These cancel, leaving $H_0 t_0 = 2/3$ as needed.

Third, what if $\Omega_0 \to 0$, where we expect $t_0 \to \infty$ logarithmically? If in fact $\Omega_0 = \epsilon \ll 1$, then to lowest order the equation gives

$$H_0 t_0 = \frac{2}{3} \ln(2/\sqrt{\epsilon}) \ . \tag{10}$$

Note that, for example, we write $1/\sqrt{1 - \Omega_0} = 1/\sqrt{1 - \epsilon} \approx 1$ to lowest order. This does indeed go to infinity logarithmically, so it satisfies our expectations.

Intuition Builder

We have ignored radiation and relativistic matter throughout. How much of a difference would it make to the estimated age if we included their effects?